Software Engineering

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Software Engineering – Verification using formal methods

Part I: Sequential programs

1. INTRODUCTION

Code should be « **working** » not only « **running** »

- This is why testing was introduced
- Testing is good and necessary, but it has limitations

Limitation of testing #1

- 100 % test coverage is out of reach

  too many lines of code,
  too many branches,
  parallelism,
  …

⇒ Many bugs may survive the testing phase

- The probability of a rare bug to occur during the software lifetime may be far above the probability that it occurs during testing

Limitation of testing #2

- Tested program may behave unpredictably
- Several possible causes:
  - Variable execution environment
    e.g., compiler, architecture, load, …
  - Unpredictable effect of uncaught programming errors
    e.g., use of non-initialized variable, div-by-0, …
  - Intrinsic program nondeterminism
    (same input => different output)
    e.g., parallel systems (variable communication delays, asynchrony)
- It is difficult/impossible to test all situations

Example (1/3)

Test the following C program

```
int main () {
      int x = 1;
      x = x++;
      assert (x == 2);
}
```

Example (2/3)

- Tested on Linux iX86 with Gnu CC 4.4.5 compiler: test passes
- Test is **exhaustive** and **successful**!
- Program can thus be safely deployed in the customer environment… Really?

Example (3/3)

- Customer uses 32-bit SunCC/Solaris compiler
- Assertion is violated: x == 1
- Cause: ambiguity of x = x++, unspecified order between assignment x = … and increment x++

/* x == 1 */ R = x; (read x) x = R + 1; (Increment x) x = R; (assign x) /* x == 1 */

vs.

/* x == 1 */ R = x; (read x) x = R; (assign x) x = R + 1; (increment x) /* x == 2 */

where R is a register used to store the initial value of x

General problem: improve predictability

- Motivation: errors are costly

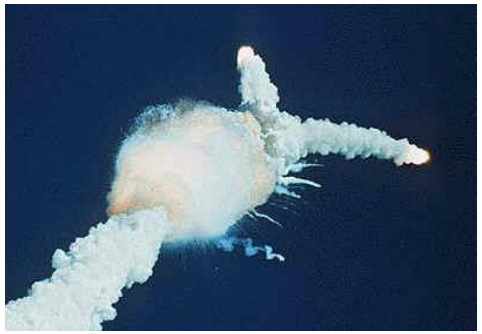
  « Normal » software:

  Critical software:




  e.g., avionics, aerospace, automotive, nuclear, chemicals, …

- Cost increases over time!
- Verification methods complementary to test are needed to find bugs early



How to improve predictability?

- Use « clean » programming languages:
  - Static semantic checks to avoid common errors (uninitialized variable, division by 0, etc.)
  - Well-defined semantics
- But this is not enough to ensure that programs will always provide a correct result:
  - Need to describe the programmer's intent: logic
  - Need to determine how intent is achieved by the program: reasoning

Models of programs

- Programming languages are sometimes too low-level for formal reasoning
- Rather use **models** of **higher abstraction level**
  - Abstract away from implementation details to focus on algorithmic problems
  - Helps getting convinced of correctness
- Models are a generalization of programs, with additional abstraction mechanisms
  - Example: **nondeterminism** used to underspecify parts that are not essential to correctness
- In the sequel: program = model

Formal verification methods

- Formal = mathematically defined
- Relies on formal languages to model:
  - Programs
  - Requirements
- **Advantages**:
  - Eliminate the risk of ambiguities
  - Offer mathematically based (rigorous) verification methods

Formal verification

- Several formal verification methods exist, with many criteria of choice
- One major criteria is whether the program is *transformational* or *reactive/concurrent*

This lecture:

- Week 10: transformational programs – proof techniques
- Week 11: reactive & concurrent programs – automata-based verification

2. PROVING THE CORRECTNESS OF TRANSFORMATIONAL PROGRAMS

Transformational program

- Program (or part of a program)
  - Computes an output in function of an input
  - Essentially behaves sequentially (even though implementation may be parallel)
  - Execution should terminate (otherwise error)

- Examples :
  - Imperative programs (C, C++, Ada, …)
  - Functional programs (ML, Scheme)

Proving transformational programs correct

- **Goal**: ensure that program behaves *as expected*
- Several possible notions of *as expected*
  - Absence of crash: No unexpected termination
    Examples: division by zero, out-of-bound array access, etc.
  - Correctness: A particular relation between program inputs and outputs holds
  - Termination: No infinite execution
  - Performance: Bounded usage of resources (e.g., time, memory, etc.)
- This lecture focuses essentially on program correctness

Program correctness

Proving a program correct requires:

- A formal **specification** of the program
- A formal **specification** of the mathematical property that the program should satisfy
- Formal deduction rules to relate property and program (reasoning)

Methods to prove specifications of transformational programs

- Algebraic methods
- Hoare logic
- Set-based methods:
  - Z, VDM
  - B: combines ideas from Z and from Hoare logic

2.1. ALGEBRAIC METHODS

Principle of algebraic methods

- Formal framework to demonstrate mathematical properties of programs
- Use of **algebra** and **equational logic**
- **Specification by properties**: objects are **defined by the operations** that generate or use them, as **mathematical equations**
- There are many algebraic specification languages (ACT ONE, LARCH, LPG, ...)

First example: Booleans

- Boolean values are written **True**/**False**; **Bool** = {**True**, **False**}
- Opns **Not**, **Or**, **And** are defined by the following equations:

Not (**True**) = **False**

Not (**False**) = **True**

($\forall$X $\in$ **Bool**) **Or** (**True**, X) = **True**

($\forall$X $\in$ **Bool**) **Or** (**False**, X) = X

($\forall$X $\in$ **Bool**) **And** (**True**, X) = X

($\forall$X $\in$ **Bool**) **And** (**False**, X) = **False**

- We call *terms* the variables, constants and operations applied (recursively) to terms

Second example: Natural numbers

- Natural numbers are written 0, **S** (0), **S** (**S** (0)), ...; **Nat** is the set of natural numbers (**S** and 0 are called ***constructors***)
- Operations **Pred** (predecessor), +, and $\times$ are defined by the following equations:

$(\forall X \in \mathbf{Nat})$	$\mathbf{Pred}\ (\mathbf{S}\ (X)) = X$	Déf. of **Pred**
$(\forall Y \in \mathbf{Nat})$	$0 + Y = Y$	Déf. of +
$(\forall X, Y \in \mathbf{Nat})$	$\mathbf{S}\ (X) + Y = \mathbf{S}\ (X + Y)$	
$(\forall Y \in \mathbf{Nat})$	$0 \times Y = 0$	Déf. of $\times$
$(\forall X, Y \in \mathbf{Nat})$	$\mathbf{S}\ (X) \times Y = Y + (X \times Y)$	

Example of proof (1)

Prove that **S** (**S** (0)) $\times$ **S** (**S** (0)) = **Pred** (**S** (**S** (**S** (**S** (**S** (0))))))

Simple application of the equations:

S (**S** (0)) $\times$ **S** (**S** (0))

= **S** (**S** (0)) + (**S** (0) $\times$ **S** (**S** (0)))	from ($\forall$X, Y$\in$**Nat**) **S** (X) $\times$ Y = Y + (X $\times$ Y)
= **S** (**S** (0)) + (**S** (**S** (0))+(0$\times$**S** (**S** (0))))	from ($\forall$X, Y$\in$**Nat**) **S** (X)$\times$Y = Y+(X$\times$Y)
= **S** (**S** (0)) + (**S** (**S** (0)) + 0)	from ($\forall$Y $\in$ **Nat**) 0 $\times$ Y = 0
= **S** (**S** (0) + **S** (**S** (0)))	from ($\forall$X, Y$\in$**Nat**) **S** (X)+Y = **S** (X+Y)
= **S** (**S** (0 + **S** (**S** (0)))	from ($\forall$X, Y$\in$**Nat**) **S** (X)+Y = **S** (X+Y)
= **S** (**S** (**S** (**S** (0))))	from ($\forall$Y $\in$ **Nat**) 0 + Y = Y
= **Pred** (**S** (**S** (**S** (**S** (**S** (0))))))	from ($\forall$X $\in$ **Nat**) **Pred** (**S** (X)) = X

Example of proof (2)

- Prove that ($\forall X \in$ **nat**) X + 0 = X
- ***By structural induction on X:*** **Base case** X = 0.

X + 0 = 0 + 0	from the hypothesis X = 0
= 0	from the equation ($\forall Y \in$ **nat**) 0 + Y = Y
= X	from the hypothesis X = 0

- **Inductive case** suppose that ($\exists$ X’ $\in$ **nat**) X' + 0 = X‘ (***induction hypothesis***) and consider X = **S** (X').

X + 0	= **S** (X') + 0	from the hypothesis X = **S** (X')
	= **S** (X' + 0)	from ($\forall$X, Y $\in$ **nat**) **S**(X) + Y = **S**(X+Y)
	= **S** (X')	from the ***induction hypothesis***
	= X	from the hypothesis X = **S** (X‘)

Third example: factorial

- The factorial operation **fact** (X) of a natural number X can be defined by the following equations:

$$\mathbf{S}\,(0) = \mathbf{fact}\,(0)$$

$$(\forall X \in \mathbf{Nat}) \qquad \mathbf{S}\,(X) \times \mathbf{fact}\,(X) = \mathbf{fact}\,(\mathbf{S}\,(X))$$

Example of proof

- Prove that

$$(\forall X \in \textbf{nat})\ X \neq 0 \Rightarrow \textbf{fact}\ (X) = X \times \textbf{fact}\ (\textbf{Pred}\ (X))$$

- Suppose $X \neq 0$. Then there exists X' such that $X = \textbf{S}\ (X')$.

$$\begin{aligned} \textbf{fact}\ (X) &= \textbf{fact}\ (\textbf{S}\ (X')) && (1)\\ &= \textbf{S}\ (X') \times \textbf{fact}\ (X') && (2)\\ &= \textbf{S}\ (X') \times \textbf{fact}\ (\textbf{Pred}\ (\textbf{S}\ (X'))) && (3)\\ &= X \times \textbf{fact}\ (\textbf{Pred}\ (X)) && (1) \end{aligned}$$

(1) from $X = \textbf{S}\ (X')$

(2) from $(\forall X \in \textbf{Nat})\ \textbf{S}\ (X) \times \textbf{fact}\ (X) = \textbf{fact}\ (\textbf{S}\ (X))$

(3) from $(\forall X \in \textbf{Nat})\ \textbf{Pred}\ (S\ (X)) = X$

Exercise

- A type **nat_list** representing lists of natural numbers is defined using the constructors **Nil**: $\rightarrow$ **nat_list** and **Cons**: **nat**, **nat_list** $\rightarrow$ **nat_list**
- Define the operation **last**: **nat_list** $\rightarrow$ **nat**, which returns the last element of a non-empty list
- Define the operation **append** : **nat**, **nat_list** $\rightarrow$ **nat_list** which appends an element at the end of a list
- Show that:
  $(\forall X \in$ **nat**, $L \in$ **nat_list**$)$ **last** (**append** $(X, L)) = X$

Solution (1/2)

- **last** : **nat_list** $\rightarrow$ **nat**

  $(\forall$ X $\in$ **nat**) **last** (**Cons** (X, **Nil**)) = X

  $(\forall$ X, Y $\in$ **nat**, L $\in$ **nat_list**)

  last (**Cons** (X, **Cons** (Y, L))) = **last** (**Cons** (Y, L))

- **append** : **nat**, **nat_list** $\rightarrow$ **nat_list**

  $(\forall$ X $\in$ **nat**) **append** (X, **Nil**) = **Cons** (X, **Nil**)

  $(\forall$ X, Y $\in$ **nat**, L $\in$ **nat_list**)

  append (X, **Cons** (Y, L)) = **Cons** (Y, **append** (X, L))

Solution (2/2)

We first prove the following **Lemma** :
($\forall$X$\in$ **nat**, L $\in$ **nat_list**) ($\exists$ Y $\in$ **nat**, L' $\in$ **nat_list**) **append** (X, L) = **Cons** (Y, L')

Proof: By case on L (immediate from the definition of **append**)

We now prove by induction on L :
($\forall$X$\in$ **nat**, L $\in$ **nat_list**) **last** (**append** (X, L)) = X

- **Base case** : L = **Nil**
  last (**append** (X, **Nil**)) = **last** (**Cons** (X, **Nil**)) = X
- **Inductive case** : Assume that for some L, **last** (**append** (X, L)) = X
  and show that for L' = **Cons** (Y, L), **last** (**append** (X, L')) = X

last (**append** (X, L')) = **last** (**append** (X, **Cons** (Y, L)))
= **last** (**Cons** (Y, **append** (X, L))) by def. of **append**
= **last** (**Cons** (Y, **Cons** (Y', L''))) **for some** Y', L'' (**Lemma**)
= **last** (**Cons** (Y', L''))) by def. of **last**
= **last** (**append** (X, L))
= X by **induction hypothesis**

Automated proofs

- In general, it is **not possible** to automate proofs
- In specific cases, proofs can be made automatic by orienting the equations in the form of ***rewriting rules***

  Example :

  fact (0) $\rightarrow$ S (0)

  fact (**S** (X)) $\rightarrow$ **S** (X) x **fact** (X)
- Certain conditions (***confluence*** and ***termination*** of the rewriting rules) must be fulfilled for the proof to be automated
- Confluence and termination of the rewriting rules cannot be proven automatically in general

Algebraic languages

Many algebraic languages exist:

- **OBJ** (J. Goguen - Université de Californie, USA - 1976)
  Ancestor of many dialects : OBJ3, CafeOBJ, BOBJ, ...
- **Larch** (J. Wing - MIT, Massachussets, USA – 1983)
- **ACT ONE** (H. Ehrig - Technische Universität Berlin, Allemagne – 1983)
  Reused in the LOTOS process algebra, followed by ACT TWO
- **PLUSS** (M.-C. Gaudel - Université Paris Sud, France – 1984)
- **LPG** (D. Bert et R. Echahed - Grenoble, France - 1984, révision 1991)
- **CASL** (*Common Algebraic Specification Language* - Initiative of a group of researchers from several origins, 1997)
- etc.

Software tools for algebraic methods

- **Larch prover** (MIT, Massachussets, USA) www.sds.lcs.mit.edu/spd/larch

  Interactive prover for Larch (maintained but not anymore developed)
- **Maude** (SRI, Californie, USA) www.csl.sri.com/projects/maude

  Rewrite engine based on an inheritor of OBJ
- **Elan** (INRIA, Nancy, France) http://elan.loria.fr

  Rewrite engine
- **CASL consistency checker** (Bremen University, Germany) www.informatik.uni-bremen.de/cofi

  Verification of the consistency of a specification
- **ACL2** (Texas University, USA) www.cs.utexas.edu/users/moore/acl2

  Theorem prover by rewriting based on a dialect of Lisp

  Used by AMD to verify correctness of elementary operations on the floating point numbers of the Athlon processor

  Laureat in 2005 of the "ACM Software System Award"

Conclusion on algebraic specifications

- Algebraic specifications are a formal framework to reason on transformational programs

But

- Writing algebraic specifications is hard
  - One does not always know whether enough equations have been written to fully model the program (***completeness***)
  - One does not always know whether contradicting equations have been written (***consistency***)
  - Completeness and consistency cannot be proven automatically
- Algebraic specifications are not executable if they are not oriented as rewriting rules
- Algebraic specifications languages are not well adapted to express the notion of ***program state***

Exercise (1/4)

Consider the following specification of Booleans and lists of Booleans:

Definition : true : $\rightarrow$ Bool false : $\rightarrow$ Bool and : Bool × Bool $\rightarrow$ Bool

for all $b \in$ Bool:

(*a1*) true and $b = b$

(*a2*) false and b = false

Definition: nil : $\rightarrow$ Bool_List cons : Bool × Bool_List $\rightarrow$ Bool_List

cat : Bool_List × Bool_List $\rightarrow$ Bool_List

for all $b \in$ Bool, $l_0, l_1, l_2 \in$ Bool_List:

(*c1*) cat (nil, l_0) = l_0

(*c2*) cat (cons (b, l_1), l_2) = cons (b, cat (l_1, l_2))

Definition: and_list : Bool_List $\rightarrow$ Bool

for all $b \in$ Bool, $l \in$ Bool_List:

(*l1*) and_list (nil) = true

(*l2*) and_list (cons (b, l)) = b and and_list (l)

We propose to show that for all $l_1, l_2 \in$ Bool_List:

(*eqn*) and_list (cat (l_1, l_2)) = and_list (l_1) and and_ list (l_2)

Exercise (2/4)

What does the cat operation do?

Exercise (2/4)

What does the cat operation do?

Concatenation of lists

example: cat ({true, false}, {true})

cat (cons (true, cons (false, nil)), cons (true, nil))

= cons (true, cat (cons (false, nil), cons (true, nil)) (*c2*)

= cons (true, cons (false, cat (nil, cons (true, nil)))) (*c2*)

= cons (true, cons (false, cons (true, nil))) (*c1*)

{true, false, true}

Exercise (3/4)

In (*eqn*), consider the particular case where
l_1 = nil and l_2 is an arbitrary list

complete the following lines:

and_list (cat (nil, l_2)) = . . . by (*c1*)

and_list (nil) and and_list (l_2)

= . . . by (*l1*)

= . . . by (*a1*)

What can you conclude about (*eqn*) when l_1 = nil?

Exercise (3/4)

In (*eqn*), consider the particular case where
l_1 = nil and l_2 is an arbitrary list

complete the following lines:

and_list (cat (nil, l_2)) = and_list (l_2) by (*c1*)

and_list (nil) and and_list (l_2)

= true and and_list (l_2) by (*l1*)

= and_list (l_2) by (*a1*)

What can you conclude about (*eqn*) when l_1 = nil?

It holds: and_list (cat (*nil*, l_2)) = and_list (*nil*) and and_list (l_2)

Exercise (4/4)

We now assume that there exists at least one list $l_3 \in$ Bool_List such that (*eqn*) holds, i.e., for all $l_2 \in$ Bool_List:

(*ih*) and_list (cat (l_3, l_2)) = and_list (l_3) and and_list (l_2)

We consider the list l_4 = cons (b, l_3) where b is an arbitrary Boolean, and we then show the following using (*ih*):

for all *l2* $\in$ Bool_List:

and_list (cat (*l4*, *l2*)) = and_list (*l4*) and and_list (*l2*)

(The proof is not asked)

What can we conclude about (*eqn*)?

How is called this kind of reasoning?

What do the initials *ih* stand for?

Exercise (4/4)

We now assume that there exists at least one list $l_3 \in$ Bool_List such that (*eqn*) holds, i.e., for all $l_2 \in$ Bool_List:

(*ih*) and_list (cat (l_3, l_2)) = and_list (l_3) and and_list (l_2)

We consider the list l_4 = cons (*b*, l_3) where *b* is an arbitrary Boolean, and we then show the following using (*ih*):

for all *l2* $\in$ Bool_List:

and_list (cat (*l4*, *l2*)) = and_list (*l4*) and and_list (*l2*)

(The proof is not asked)

What can we conclude about (*eqn*)? holds for any l_1 : nil or cons (...)

How is called this kind of reasoning? reasoning by induction

What do the initials *ih* stand for? induction hypothesis

2.2. HOARE LOGIC AND DESIGN BY CONTRACT

Hoare logic

- Seen earlier in the « compiler » class
- A framework for proving programs, proposed by Tony Hoare in 1969, inspired by Robert Floyd
- Mathematical formalization of deduction rules for reasoning on programs
- Motivations:
  - Rigorous definition of reasoning (teaching, research papers, …)
  - Implementation in tools

Reminder about Hoare logic

- Hoare triples { P } C { Q } where P, Q are assertions in first-order predicate logic, called **precondition** and **postcondition**
- **Meaning**: If P holds before executing C, then Q holds after executing C
- Hoare logic is about proving Hoare triples
- Proof requires additional user-given assertions called **loop variants** and **loop invariants**
- Many systems for proving sequential programs somehow rely on extensions of Hoare logic
  Example: The B method

Reminder about first-order predicate logic

- ***Terms*** represent data: constants, variables, function applications

  Examples : x, 7, **true**, **false**, **sin**(x), x < y, y + 1

- ***Formulas*** may take several forms:
  - ***Predicates***: terms that evaluate to true or false

    Examples: **true**, **false**, **even** (x), x < y, etc.
  - ***Propositional formulas***: built using predicates and the logic connectors $\wedge$, $\vee$, $\Rightarrow$, $\neg$, etc.

    Examples : f (x, y) $\wedge$ f (y, z) $\Rightarrow$ f (x, z)
  - ***Quantified formulas***: with logic quantifiers ($\forall$ x . A), ($\exists$ x . A)

Programming with Hoare logic assertions: design by contract

- A methodology proposed by B. Meyer (1986) and first implemented in the Eiffel programming language
- Write contract (*what should be done*) together with code (*how this is done*):
  - A pre and a postcondition with each function
  - An invariant and a variant with each loop
  - In OO programming, a property on state variables that should hold before and after every method call, named object invariant
- Contracts can be checked at runtime or, if the programming and assertion languages have formal semantics, connexion to provers is possible

Contracts in SPEC#

SPEC# (Microsoft Research)
http://research.microsoft.com/en-us/projects/specsharp

- Formal language for contracts
- Extends C#, integrated in Visual Studio
- Boogie program verifier connected to automatic prover of logic properties
- Web interface (for toy examples): http://rise4fun.com/SpecSharp
- **Homework**: watch http://channel9.msdn.com/Blogs/Peli/The-Verification-Corner-Specifications-in-Action-with-SpecSharp

Factorial in SPEC#

```
class Factorial {
    static int fact (int n)
    requires n >= 0;
    ensures
      result == product{int i in (1:n+1); i};
      /* product of ints from 1 to n */
    {
        int x, r;
        x = n;
        r = 1;
    }
```

```
while (x > 0)
        invariant x >= 0;
        invariant x <= n;
        invariant
          r == product{int i in (x+1:n+1); i};
          /* product of ints from x+1 to n */
        {
            r *= x;
            x--;
        }
        return r;
    }
```

Conclusion on Hoare logic and contracts

Beyond formal proof, assertions radically change the nature of software development in several ways:

- **Design aid**: build program + arguments that justify its correctness
- **Testing and debugging**: assertions can be checked at runtime
- **Documentation**: non-ambiguous and concise description of what the program does (instead of how this is done)

2.3. SET-BASED METHODS

Set-based specification languages

- They are formal languages appropriate to describe:
  - The notion of **program state**, defined by a set of typed **variables**
  - Program **operations**, defined by their inputs, their outputs, an **application condition** (**precondition**) and an **effect** on the state variables
- They use set-based notations and first-order predicate logic
- They generally come with a method that defines good development practices and rely on software tools

Main set-based methods

- **VDM** (*Vienna Development Method*)
  - Ancestor of set-based methods
  - Invented at the Vienna IBM laboratory in the 60's
  - Used by some industries including DCC-International (Ada compiler), British Aerospace, Adelard, ...
- **The Z notation**
  - Set-based notation proposed par J.-R. Abrial (1977) and developed by the team of Tony Hoare in Oxford
  - Standardized at ISO in 2002
- **The B method**
  - Method proposed by J.-R. Abrial in the 80's

The B method

- Method based on ***Abstract State Machines***, which unify the notions of (set-based) specification, proof and executable code
- With industrial usages:
  - **Matra Transport** and **RATP** : verification of the control system for safety equipments of Paris metro line 14 (automatic trains), from specification to Ada code generation
  - But also: **Gemalto** (smart cards), **Siemens**, **Leirios Technologies**, ...

B: Methodology

Seamless methodology from specification to executable code:

1. Program specification as an abstract state machine
2. Automatic generation of properties to be proven for the satisfaction of variants and invariants: the ***proof obligations***
3. ***Progressive refinement*** of the state machine : Manual replacement of non-executable elements by executable ones
4. Automatic generation and proof of new **proof obligations** that express the preservation of properties proven at the previous step
5. Back to point 3 until obtention of executable code

B: ASM (Abstract State Machine)

It is defined by:

- Its **name**
- Its **state variables**
- A **state invariant** (formula of first-order predicate logic): property that must be true at initialisation and remain true after each application of an operation
- A **variable initialisation** clause
- A list of **operations** that read **inputs**, return **outputs** and modify the state

ASM example: Plane boarding system (1/2)

MACHINE Plane

SEES Bool_TYPE

Imported type

SETS PASSENGERS

CONSTANTS cap

PROPERTIES $cap \in N$

Property of constants

VARIABLES onboard

INVARIANT

$(onboard \subseteq PASSENGERS) \wedge (card\ (onboard) \leq cap)$

INITIALISATION

$onboard := \varnothing$

Generalised substitution (variable assignment)

OPERATIONS

.../...

ASM example: Plane boarding system (2/2)

…/…

OPERATIONS

Boarding (p) =
PRE (p ∈ PASSENGERS) ∧ (p ∉ onboard)
THEN onboard := onboard ∪ { p }
END;

b ← Onboard (p) =
PRE p ∈ PASSENGERS
THEN **IF** p ∈ onboard
THEN b := TRUE
ELSE b := FALSE
END
END

END

B: Operations

An operation has the following form:

outputs ← op (***inputs***) =
PRE
precondition
THEN
effect
END

where:

- **precondition** is a formula of first-order predicate logic
- **inputs**, **outputs** are lists of local variables
- **effect** is a statement (generalised substitution)

Examples of generalised substitutions

- Deterministic assignment: X := E
- Nondeterministic assignment: X :$\in$ T

  X takes any value in the set T
- Sequential composition: S_1; S_2

  substitution S_1 followed by S_2
- Conditional branch: **IF** P **THEN** S_1 **ELSE** S_2 **END**
- Nondeterministic branch: **CHOICE** S_1 **OR** … **OR** S_n **END**

  Arbitrary choice among S_1, …, S_n
- Loop**: WHILE** P **DO** S_0 **VARIANT** E **INVARIANT** Q **END**
- etc.

Proof obligations

- The B method defines first-order predicate logic formulas that must be proven for the invariant to hold: the **proof obligations**
- They are calculated by applying a substitution S to a formula Q, written "[S] Q"
- Relationship with Hoare logic: [S] Q = the (weakest) precondition that S must satisfy for postcondition Q to be satisfied
- Proving "P $\Rightarrow$ [S] Q" is thus analoguous to proving the goal { P } S { Q }, but with a generalisation to nondeterministic statements

Application of substitutions (1/3)

- $[\ X := E\ ]\ Q \equiv Q$ in which E replaces X

  Example : $[\ x := 1\ ]\ (x \leq c) \equiv (1 \leq c)$

- $[\ X :\in S\ ]\ Q = (\forall y\ .\ y \in S \Rightarrow [\ X := y\ ]\ Q)$

  Example : $[\ x :\in N\ ]\ p(x) \equiv$

  $(\forall y\ .\ y \in N \Rightarrow p(y)) \equiv$

  $p(0) \wedge p(1) \wedge p(2) \wedge \ldots$

- [**CHOICE** S_1 **OR** S_2 **END**] $Q \equiv ([\ S_1\ ]\ Q) \wedge ([\ S_2\ ]\ Q)$

Application of substitutions (2/3)

- [**IF** P **THEN** S_1 **ELSE** S_2 **END**] Q ≡

  $(P \Rightarrow [\ S_1\ ]\ Q) \wedge (\neg P \Rightarrow [\ S_2\ ]\ Q)$

  Example:

  [**IF** x > 0 **THEN** x := x-1 **ELSE** x := x+1 **END**] p(x) ≡

  $(x > 0 \Rightarrow p(x-1)) \wedge (x \leq 0 \Rightarrow p(x+1))$

- $[\ S_1; S_2\ ]\ Q \equiv [\ S_1\ ]\ [\ S_2\ ]\ Q$

  Example: [x := 2 * z; y := x+1] p(y) ≡

  [x := 2 * z] p(x+1) ≡

  p((2 * z) + 1)

Application of substitutions (3/3)

- [**WHILE** P **DO** S_0 **INVARIANT** I **END**] Q ≡

$$I \wedge (\forall \mathbf{X})\ (I \wedge P \Rightarrow [S_0]\ I) \wedge (\forall \mathbf{X})\ (I \wedge \neg P \Rightarrow Q)$$

(we omit the **VARIANT** part here for simplification)

Example:

[**WHILE** $X \neq N$ **DO** $Y := Y+X;\ X := X+1$
INVARIANT $Y = \Sigma_{i \in 0..X-1}\ i$ **END**] $(Y = \Sigma_{i \in 0..N-1}) \equiv$

$$Y = \Sigma_{i \in 0..X-1}\ i$$

$$\wedge$$

$$(\forall X,Y,N)\ (Y = \Sigma_{i \in 0..X-1}\ i \wedge X \neq N \Rightarrow [Y:=Y+X;\ X:=X+1\ ]\ Y = \Sigma_{i \in 0..X-1}\ i)$$

$$\wedge$$

$$(\forall X,Y,N)\ (Y = \Sigma_{i \in 0..X-1}\ i \wedge X = N \Rightarrow Y = \Sigma_{i \in 0..N-1})$$

Exercise

Compute the following substitution

[**IF** X < Y **THEN** MIN := X **ELSE** MIN := Y **END**] (MIN = X)

Solution

[**IF** X < Y **THEN** MIN := X **ELSE** MIN := Y **END**] (MIN = X) ≡

(*Applying IF-THEN-ELSE substitution*)

(X < Y ⇒ [MIN := X] (MIN = X)) ∧ (X ≥ Y ⇒ [MIN := Y] (MIN = X)) ≡

(*Applying assignment substitutions*)

(X < Y ⇒ X = X) ∧ (X ≥ Y ⇒ X = Y) ≡

(*Replacing* X = X *by* **true**)

(X < Y ⇒ **true**) ∧ (X ≥ Y ⇒ X = Y) ≡

(*Replacing* P ⇒ **true** *by* **true**)

true ∧ (X ≥ Y ⇒ X = Y) ≡

(*Replacing* **true** ∧ P *by* P)

X ≥ Y ⇒ X = Y ≡

(*Replacing* P ⇒ Q *by* ¬P ∨ Q)

X < Y ∨ X = Y

Exercise

Compute the following expression

$$Y = \Sigma_{i \in 0..X-1}\, i \wedge X \neq N \Rightarrow [Y:=Y+X;\ X:=X+1\ ]\ Y = \Sigma_{i \in 0..X-1}\, i$$

and check that it is true

Solution

$$Y = \Sigma_{i\in 0..X-1}\, i \wedge X \neq N \Rightarrow [Y:=Y+X;\ X:=X+1\ ]\ Y = \Sigma_{i\in 0..X-1}\, i \equiv$$

$$Y = \Sigma_{i\in 0..X-1}\, i \wedge X \neq N \Rightarrow [Y:=Y+X]\ Y = \Sigma_{i\in 0..X+1-1}\, i \equiv$$

$$Y = \Sigma_{i\in 0..X-1}\, i \wedge X \neq N \Rightarrow [Y:=Y+X]\ Y = \Sigma_{i\in 0..X}\, i \equiv$$

$$Y = \Sigma_{i\in 0..X-1}\, i \wedge X \neq N \Rightarrow Y+X = \Sigma_{i\in 0..X}\, i \equiv$$

$$Y = \Sigma_{i\in 0..X-1}\, i \wedge X \neq N \Rightarrow Y = \Sigma_{i\in 0..X}\, i - X \equiv$$

$$Y = \Sigma_{i\in 0..X-1}\, i \wedge X \neq N \Rightarrow Y = \Sigma_{i\in 0..X-1}\, i \equiv$$

true

Proof obligations

We write INV for the invariant of the state machine

- The substitution INIT which initialises the variables must **establish the state invariant**

  $\Rightarrow$ proof obligation [INIT] INV

  Analogy in Hoare logic: { true } INIT { INV }

- Each operation

  $$outputs \leftarrow op\ (inputs) = \textbf{PRE}\ P\ \textbf{THEN}\ S\ \textbf{END}$$

  must **preserve the state invariant**

  $\Rightarrow$ proof obligation (INV $\wedge$ P) $\Rightarrow$ [S] INV

  Analogy in Hoare logic: { INV $\wedge$ P } S { INV }

Proof obligation examples (1/2)

PROPERTIES $cap \in N$

INVARIANT

$(onboard \subseteq PASSENGERS) \wedge (card\ (onboard) \leq cap)$

INITIALISATION

$onboard := \varnothing$

Proof obligation (initialisation) :

$[\ onboard := \varnothing\ ]\ (onboard \subseteq PASSENGERS) \wedge (card\ (onboard) \leq cap) \equiv$

$(\varnothing \subseteq PASSENGERS) \wedge (card\ (\varnothing) \leq cap) \equiv$

true CQFD

Proof obligation examples (2/2)

Boarding (p) =

PRE $(p \in PASSENGERS) \wedge (p \notin onboard)$
THEN $onboard := onboard \cup \{ p \}$
END

Proof obligation:

$(p \in PASSENGERS) \wedge (p \notin onboard) \wedge$
$(onboard \subseteq PASSENGERS) \wedge (card\ (onboard) \leq cap)$
$\Rightarrow$
$(onboard \cup \{ p \} \subseteq PASSENGERS) \wedge$
$(card\ (onboard \cup \{ p \}) \leq cap)$

Not provable : ERROR if $card\ (onboard) = cap$

Conclusion: The specification is incorrect, we must add the missing precondition: $card\ (onboard) < cap$

B example: Voting machine (1/2)

- We specify in B a simplified voting machine that records the votes for 2 candidates, represented by the numbers 1 and 2.
- The variables votes1 and votes2 hold the number of votes for each of the candidates
- The variable cast holds the voters who have already voted

B example: Voting machine (2/2)

MACHINE Voting_Machine
SETS VOTERS
VARIABLES votes1 votes2 cast
INVARIANT votes1 $\in \mathbb{N} \wedge$ votes2 $\in \mathbb{N} \wedge$ cast $\subseteq$ VOTERS $\wedge$
card (cast) = votes1 + votes2
INITIALISATION votes1 := 0; votes2 := 0; cast := $\varnothing$
OPERATIONS
Vote (e, n) = **PRE** e $\in$ VOTERS $\wedge$ n $\in$ {1, 2} **THEN**
cast := cast $\cup$ {e};
IF n = 1 **THEN**
votes1 := votes1 + 1
ELSE
votes2 := votes2 + 1
END
END
END

Question n°1

- What is the proof obligation that would allow to guarantee that the initialisation establishes the invariant?
- Explain the main steps of the computation
- Explain (even informally) why this proof obligation is true or false

Response to question n°1

- For an invariant INV and an initialisation substitution INIT, the proof obligation is [INIT] INV. Here:

  [votes1 := 0; votes2 := 0; cast := ∅] (votes1 ∈ ℕ ∧ votes2 ∈ ℕ ∧ cast ⊆ VOTERS ∧ card (cast) = votes1 + votes2)

  ≡ 0 ∈ ℕ ∧ 0 ∈ ℕ ∧ ∅ ⊆ VOTERS ∧ card (∅) = 0 + 0

- This property is true because:
  - 0 is indeed a natural number
  - The empty set is indeed a subset of VOTERS (since it is a subset of any set)
  - The cardinal of the empty set is indeed 0

Question n°2

- What is the proof obligation that would allow to guarantee that the operation Vote preserves the invariant?
- Explain the main steps of the computation
- Explain (even informally) why this proof obligation is true or false

Response to question n°2 (1/2)

- For an invariant INV, a precondition PRE and a substitution SUB, the proof obligation is
  INV $\wedge$ PRE $\Rightarrow$ [SUB] INV. Here:

  votes1 $\in \mathbb{N} \wedge$ votes2 $\in \mathbb{N} \wedge$ cast $\subseteq$ VOTERS $\wedge$ card (cast) = votes1 + votes2 $\wedge$ e $\in$ VOTERS $\wedge$ n $\in$ {1, 2}

  $\Rightarrow$

  [cast := cast $\cup$ {e}; **IF** n = 1 **THEN** votes1 := votes1 + 1 **ELSE** votes2 := votes2 + 1 **END**] (votes1 $\in \mathbb{N} \wedge$ votes2 $\in \mathbb{N} \wedge$ cast $\subseteq$ VOTERS $\wedge$ card (cast) = votes1 + votes2)

- The right-hand-side [SUB] INV of the implication yields:

  (n = 1 $\Rightarrow$ (votes1 + 1) $\in \mathbb{N} \wedge$ votes2 $\in \mathbb{N} \wedge$ cast $\cup$ {e} $\subseteq$ VOTERS $\wedge$ card (cast $\cup$ {e}) = votes1 + 1 + votes2) $\wedge$

  ($\neg$(n = 1) $\Rightarrow$ votes1 $\in \mathbb{N} \wedge$ (votes2 + 1) $\in \mathbb{N} \wedge$ cast $\cup$ {e} $\subseteq$ VOTERS $\wedge$ card (cast $\cup$ {e}) = votes1 + votes2 + 1)

Response to question n°2 (2/2)

- The proof obligation INV $\wedge$ PRE $\Rightarrow$ [SUB] INV cannot be proven
- One should indeed always have:
  card (cast $\cup$ {e}) = card (cast) + 1
- But if e $\in$ cast then cast $\cup$ {e} = cast, i.e., card (cast $\cup$ {e}) = card (cast) !
- For the invariant to be preserved, one should for instance strengthen the precondition of the Vote operation, to ensure that the voter has not yet cast his/her vote:
  e $\in$ VOTERS $\wedge$ e $\notin$ cast $\wedge$ n $\in$ {1, 2}

ASM refinement

- **Goal**: Transform the formal specification to executable code
- Successive manual modifications of the specification
  - Suppression of **non-executable elements**: preconditions, simultaneity, nondeterminism
  - Introduction of **control structures**
  - Transformation of **abstract data structures** (sets, relations, ...) into **programmable data structures** (arrays, files, ...)

Example : Refinement of the plane (1)

Refinement of the plane in which a seat is assigned to each passenger admitted on board

MACHINE Plane_seats

REFINES Plane

SEES Bool_TYPE

SETS SEATS = 1..cap

VARIABLES assign

INVARIANT

$(assign \in SEATS \mathrel{+\!\!\rightarrow} PASSENGERS) \wedge (onboard = rng\ (assign))$

assign: partial function that assigns a passenger to each occupied seat

INITIALISATION $assign := \varnothing$

initially, no seat is attributed

rng : image of a function (here, the set of passengers on board)

Example : Refinement of the plane (2)

Set of values on which the partial function is defined

OPERATIONS

Boarding (p) =

PRE $(p \in PASSENGERS) \wedge (dom(assign) \subset SEATS) \wedge (p \notin rng(assign))$

THEN

ANY x WHERE $x :\in (SEATS \setminus dom(assign))$

THEN assign (x) := p

END

END;

...

END

Extension of the function

Proof obligations of refinement

- The B method defines new proof obligations
  - To prove that the initialisation of the refined machine is compatible with the initialisation of the original machine
  - To prove that each operation of the refined machine is compatible with the corresponding operation of the original machine
  - And hence that the refined invariant is preserved by the refined machine
- Precise definition of these proof obligations is out of the scope of this course

Software tools for the B method

- Atelier B (http://www.atelierb.eu)
  - Commercial software developed by the company ClearSy
  - Several tools
    - Syntax analyser
    - Type controler
    - Proof obligation generator
    - Automated prover
    - Interactive prover
    - Translator into several programming languages
- Free software: JBTools, B4Free, ABTools, ProB, ...

Conclusion

- There exist formal methods to help developing reliable transformational programs
  - Set-based (or *model-based*) methods
  - Algebraic (or *property-based*) methods
- Formal methods provide many advantages : early error detection, quality and reliability, utilisability of formal specifications in the next steps of the software lifecycle (test, evolution)
- The knowledge of formal methods is **a plus in your practice of programming**

To go further (1/3)

- **Hoare logic**
  - Original article by Tony Hoare: An axiomatic basis for computer programming. CACM, 1969.
  - Wikipedia : http://en.wikipedia.org/wiki/Hoare_logic
- **VDM**
  - Cliff B. Jones. Systematic software development using VDM. Prentice Hall, 1986.
  - Wikipedia : http://en.wikipedia.org/wiki/Vienna_Development_Method
- **The Z notation**
  - J. M. Spivey. The Z notation (2nd edition). Prentice Hall, 1998. 168 pages.
  - David Lightfoot. Formal specification using Z (2nd edition). Palgrave, 2000. 176 pages.
  - Wikipedia : http://en.wikipedia.org/wiki/Z_notation

To go further (2/3)

- **The B method**
  - J.-R. Abrial. The B-Book, assigning programs to meanings. Cambridge University Press, 1996.
  - Much information and resources on B : http://www-lsr.imag.fr/B/Bsite-pages.html
  - Wikipedia : http://en.wikipedia.org/wiki/B-Method
- **Algebraic specifications**
  - H. Ehrig, B. Mahr. Fundamentals of algebraic specification. Springer, 1985. 321 pages.
  - Wikipedia : http://en.wikipedia.org/wiki/Algebraic_specification
- **Synthesis on formal methods**
  - Marc Frappier, Henri Habrias (editors). Software specification methods: an overview using a case study. Springer, 2000. 312 pages. http://www.dmi.usherb.ca/~spec
  - Wikipedia : http://en.wikipedia.org/wiki/Formal_Methods

To go further (3/3)

Tools based on higher-order languages and logics

- **PVS** (SRI, California, USA)
  - http://pvs.csl.sri.com
- **LCF** (Edinburgh, Scotland and Stanford, California, USA)
  - Ancestor of Isabelle and HOL
- **Isabelle** (Cambridge, UK and Munich, Germany)
  - http://www.cl.cam.ac.uk/research/hvg/Isabelle
- **HOL** (University of Pennsylvania, USA)
  - Acronym of *Higher Order Logic*
  - http://www.cis.upenn.edu/~hol
- **Coq** (INRIA, France)
  - http://coq.inria.fr

Competence and Knowledge which will be evaluated

- be able to
  - understand simple **algebraic specifications**, simple **abstract state machines** and **operations**
  - carry on simple **algebraic proofs**, derive and prove simple **proof obligations**
- know
  - the general notions of **precondition**, **postcondition**, **loop variant**, **loop invariant** and **state invariant**
  - **reason rigorously** on a transformational programs