Some answers for Exercise 16

1

Similarly to the rule for the While construct, we shall provide two rules depending on the value of Boolean condition b.

$$[\texttt{repeat}_{\texttt{tt}}]\ \frac{(S,\sigma)\to\sigma'}{(\texttt{repeat}\ S\ \texttt{until}\ b,\sigma)\to\sigma'}\ \mathcal{B}[b](\sigma')=\mathbf{tt}$$

$$[\texttt{repeat}_{\texttt{ff}}]\ \frac{(S,\sigma)\to\sigma' \qquad (\texttt{repeat}\ S\ \texttt{until}\ b,\sigma')\to\sigma''}{(\texttt{repeat}\ S\ \texttt{until}\ b,\sigma)\to\sigma''}\ \mathcal{B}[b](\sigma')=\mathbf{ff}$$

2

Let S_1 be $\texttt{repeat}\ S\ \texttt{until}\ b$ and S_2 be $S;\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi}$. Let us show that S_1 is semantically equivalent to S_2, that is, for all $\sigma,\sigma'\in\mathsf{State}$, $(S_1,\sigma)\to\sigma'$ if and only if $(S_2,\sigma)\to\sigma$. We shall prove the implication in both ways.

If direction Let us consider $\sigma,\sigma''\in\mathsf{State}$ and suppose $(S_1,\sigma)\to\sigma''$. According to the rules given for the repeat construct, there exists $\sigma'\in\mathsf{State}\cdot(S,\sigma)\to\sigma'$. Let us distinguish two cases, according to whether $\mathcal{B}[b](\sigma')=\mathbf{tt}$ or $\mathcal{B}[b](\sigma')=\mathbf{ff}$.

- Case $\mathcal{B}[b](\sigma')=\mathbf{tt}$. In this case, according to rule $[\texttt{repeat}_{\texttt{tt}}]$, we have $\sigma'=\sigma''$. Moreover, using rule $[\texttt{if}_{\texttt{tt}}]$, since $\mathcal{B}[b](\sigma')=\mathbf{tt}$, we have the following derivation tree:

$$\frac{(\texttt{skip},\sigma')\to\sigma'}{(\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma')\to\sigma'}$$

  Moreover, using the rule [comp] and $(S,\sigma)\to\sigma''$, the following derivation holds:

$$\frac{(S,\sigma)\to\sigma' \qquad \dfrac{(\texttt{skip},\sigma')\to\sigma'}{(\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma')\to\sigma'}}{(S;\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma)\to\sigma''}$$

- Case $\mathcal{B}[b](\sigma')=\mathbf{ff}$. Since $(S_1,\sigma)\to\sigma''$ and $(S,\sigma)\to\sigma'$ with $\mathcal{B}[b](\sigma')=\mathbf{tt}$. Moreover, using $[\texttt{repeat}_{\texttt{ff}}]$ and $(S_1,\sigma)\to\sigma''$, we have $(\texttt{repeat}\ S\ \texttt{until}\ b,\sigma')\to\sigma''$. Using rule $[\texttt{if}_{\texttt{ff}}]$, we have the following derivation tree:

$$\frac{(S_1,\sigma')\to\sigma''}{(\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma')\to\sigma''}$$

  Using rule [comp] and $(S,\sigma)\to\sigma'$, we obtain the following derivation tree:

$$\frac{(S,\sigma)\to\sigma' \qquad (\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma')\to\sigma''}{(S;\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma)\to\sigma''}$$

only if direction Let us consider $\sigma,\sigma'\in\mathsf{State}$ and suppose $(S_2,\sigma)\to\sigma'$. According to the rule [comp], there exists σ'' such that $(S,\sigma)\to\sigma''$ and $(\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma'')\to\sigma'$. Let us distinguish two cases, according to whether $\mathcal{B}[b](\sigma'')=\mathbf{tt}$ or $\mathcal{B}[b](\sigma'')=\mathbf{ff}$.

- Case $\mathcal{B}[b](\sigma'')=\mathbf{tt}$. According to the following derivation tree

$$\frac{(\texttt{skip},\sigma'')\to\sigma''}{(\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma'')\to\sigma'}\ \mathcal{B}[b](\sigma'')=\mathbf{tt}$$

  we have necessarily $\sigma''=\sigma'$ and consequently $(S,\sigma)\to\sigma'$. Moreover, $\mathcal{B}[b](\sigma'')=\mathbf{tt}$ implies $\mathcal{B}[b](\sigma')=\mathbf{tt}$. Using rule $[\texttt{repeat}_{\texttt{tt}}]$, we have the following derivation tree for $S_1=\texttt{repeat}\ S\ \texttt{until}\ b$ in σ:

$$\frac{(S,\sigma)\to\sigma'}{(\texttt{repeat}\ S\ \texttt{until}\ b,\sigma)\to\sigma'}\ \mathcal{B}[b](\sigma'')=\mathbf{tt}$$

- Case $\mathcal{B}[b](\sigma'')=\mathbf{ff}$. Since $(\texttt{if}\ b\ \texttt{then skip else}\ S_1\ \texttt{fi},\sigma'')\to\sigma'$, we have necessarily $(S_1,\sigma'')\to\sigma'$. Moreover, using $(S,\sigma)\to\sigma''$ and rule $[\texttt{repeat}_{\texttt{ff}}]$, we deduce $(\texttt{repeat}\ S\ \texttt{until}\ b,\sigma)\to\sigma'$.

3

We shall prove that the expressiveness of language While is not modified by adding the `repeat ... until ...` construct. For this purpose, we aim to provide a function $f : \mathsf{While} \to \mathsf{While}$ s.t. for all $P \in \mathsf{While}$, $f(P)$ does not contain any `repeat ... until ...` statement.
Let us consider f recursively defined on the statements of the While language:

$$
\begin{array}{l}
f(x := a) = x := a \\
f(\texttt{skip}) = \texttt{skip} \\
f(S1; S2) = f(S1); f(S2) \\
f(\texttt{if } b \texttt{ then } S1 \texttt{ else } S2 \texttt{ fi}) = \texttt{if } b \texttt{ then } f(S1) \texttt{ else } f(S2) \texttt{ fi} \\
f(\texttt{while } b \texttt{ do } S \texttt{ od}) = \texttt{while } b \texttt{ do } f(S) \texttt{ od} \\
f(\texttt{repeat } S \texttt{ until } b) = f(S); \texttt{while not } b \texttt{ do } f(S) \texttt{ od}
\end{array}
$$

We shall prove that for all $P \in \mathsf{While}$, P is semantically equivalent to $f(P)$, i.e.,

$$\text{for all } \sigma, \sigma'' \in \mathsf{State}, (P, \sigma) \to \sigma'' \text{ if and only if } (f(P), \sigma) \to \sigma''$$

The proof of this claim can be made by induction on the size of the derivation tree. Since, the function f modifies only the `repeat ... until ...` construct, we just have to prove that

$$\text{for all } \sigma, \sigma' \in \mathsf{State}, (\texttt{repeat } S \texttt{ until } b, \sigma) \to \sigma' \text{ if and only if } (S; \texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma) \to \sigma'$$

Let us remark that, according to the [`repeat`] rule, we have: there exists $\sigma' \in \mathsf{State}$ s.t. $(S, \sigma) \to \sigma'$.

if direction Let us distinguish two cases according to whether $\mathcal{B}[b](\sigma') = \mathbf{tt}$ or $\mathcal{B}[b](\sigma') = \mathbf{ff}$.

- Case $\mathcal{B}[b](\sigma') = \mathbf{tt}$. We have $(S, \sigma) \to \sigma''$ and thus $(\texttt{repeat } S \texttt{ until } b, \sigma) \to \sigma'$. Consequently $\sigma'' = \sigma'$. Since $\mathcal{B}[\texttt{not } b](\sigma') = \mathbf{ff}$, according to the rule $[\texttt{while}_{\texttt{ff}}]$, we have $(\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma'') \to \sigma''$. From $(S, \sigma) \to \sigma'$ and $(\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma'') \to \sigma''$, and using the rule [`comp`], we get

  $$(S; \texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma) \to \sigma''.$$

- Case $\mathcal{B}[b](\sigma') = \mathbf{ff}$. In this case, we perform an induction on the height of the derivation tree. The proof has been already shown for the axiom of the `while ... do ... od` . Let us suppose that the property holds for the premises of the rule. We have:

  $$\frac{(S, \sigma) \to \sigma' \qquad (\texttt{repeat } S \texttt{ until } b, \sigma') \to \sigma''}{(\texttt{repeat } S \texttt{ until } b, \sigma) \to \sigma''} \; \mathcal{B}[b](\sigma') = \mathbf{ff}.$$

  Our induction hypothesis is that the property holds for the premises of the rule and for $(\texttt{repeat } S \texttt{ until } b, \sigma') \to \sigma''$ in particular. Thus, we have:

  $$(\texttt{repeat } S \texttt{ until } b, \sigma') \to \sigma'' \text{ implies } (S; \texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma') \to \sigma''$$

  Moreover, from $(S; \texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma') \to \sigma''$ we deduce that there exists $\sigma''' \in \mathsf{State}$ s.t. $(S, \sigma') \to \sigma'''$ and $(\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma''') \to \sigma''$. Furthermore, $\mathcal{B}[b](\sigma') = \mathbf{ff}$ implies $\mathcal{B}[\texttt{not } b](\sigma') = \mathbf{tt}$ and we can deduce the following derivation tree:

  $$\frac{(S, \sigma) \to \sigma' \qquad \dfrac{(S, \sigma') \to \sigma''' \qquad (\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma'') \to \sigma''}{(\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma') \to \sigma''}}{(S; \texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma) \to \sigma''}$$

only if direction Let us suppose $(S; \texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma) \to \sigma''$. According to the [`comp`], we deduce that there exists $\sigma' \in \mathsf{State}$ s.t. $(S, \sigma) \to \sigma'$ and $(\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma') \to \sigma''$. Let us distinguish two cases according to whether $\mathcal{B}[b](\sigma') = \mathbf{tt}$ or $\mathcal{B}[b](\sigma') = \mathbf{ff}$.

- Case $\mathcal{B}[b](\sigma') = \mathbf{tt}$. From $\mathcal{B}[b](\sigma') = \mathbf{tt}$, we get $\mathcal{B}[b](\sigma') = \mathbf{ff}$, using the axiom $[\texttt{while}_{\texttt{ff}}]$ we get $(\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma'') \to \sigma''$. Consequently $\sigma' = \sigma''$. Now, using $\mathcal{B}[\texttt{not } b](\sigma') = \mathbf{ff}$ and the rule $[\texttt{repeat}_{\texttt{tt}}]$, we have the following derivation:

$$\frac{(S,\sigma) \to \sigma'}{(\texttt{repeat } S \texttt{ until } b, \sigma) \to \sigma'} \mathcal{B}[b](\sigma') = \mathbf{tt}$$

Since, $\sigma' = \sigma''$, we deduce $(\texttt{repeat } S \texttt{ until } b, \sigma) \to \sigma'$.

* Case $\mathcal{B}[b](\sigma') = \mathbf{ff}$. We use the rule $[\texttt{while}_{\texttt{ff}}]$ and deduce that there exists $\sigma''' \in \mathsf{State}$ and we have the following derivation:

$$\frac{(S,\sigma') \to \sigma''' \qquad (\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma''') \to \sigma''}{(S; \texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma') \to \sigma''}$$

Let us note $(\alpha) = (S,\sigma') \to \sigma'''$ and $(\beta) = (\texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma''') \to \sigma''$. We apply the rule [comp] to (α) and (β), and we get:

$$(S; \texttt{while not } b \texttt{ do } S \texttt{ od}, \sigma') \to \sigma' \qquad (\gamma).$$

Then, we perform an induction on height of the derivation tree. We thus suppose that the property holds for the premises of rules. Thus, from (γ) we have:

$$(\texttt{repeat } S \texttt{ until } b, \sigma') \to \sigma'.$$

Then, from $(S,\sigma) \to \sigma''$ and the previous rule, we get $(\texttt{repeat } S \texttt{ until } b, \sigma'') \to \sigma'$. Moreover, using $\mathcal{B}[b](\sigma')] = \mathbf{ff}$ and the rule $[\texttt{repeat}_{\texttt{ff}}]$, we have $(\texttt{repeat } S \texttt{ until } b, \sigma) \to \sigma'$.