INF 332: Languages & Automata

Chapter 8: Non-deterministic Finite-state Automata with ϵ-transitions

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Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

1. Motivations
2. Non-deterministic Finite-state Automata with ϵ-transitions
3. Eliminating ϵ-transitions
4. (Back to the) Closure Properties of Finite-State Languages
5. Summary

Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

1. Motivations
   - Using ϵ-transitions in Practice with Automata
   - Kleene Closure of a Language
2. Non-deterministic Finite-state Automata with ϵ-transitions
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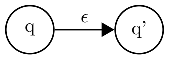
Use of ϵ-transitions

We allow ϵ as a transition label; such transitions are called ϵ-transitions.



A word is accepted if the input can be read until a final state is reached:
↪ ϵ-transitions do not "consume" any input symbols.

Underlying property:

$$\forall w \in \Sigma^* : \quad w \cdot \epsilon = \epsilon \cdot w = w.$$

(ϵ is the neutral element of word concatenation.)

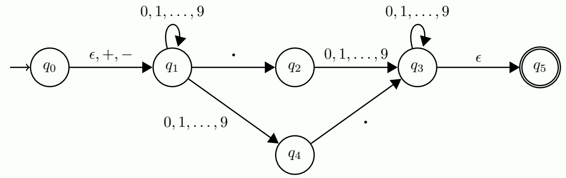
Use of ϵ-transitions

Decimal numbers

Example (Decimal numbers)

A number written in decimal notation consists of:
- an optional sign $+$ or $-$,
- a sequence of digits $0, 1, 2, \ldots, 9$,
- a decimal point,
- a sequence of digits $0, 1, 2, \ldots, 9$.

One of the two digit sequences may be empty, but not both.

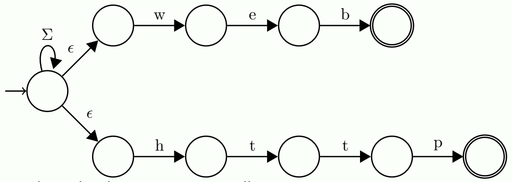


Using ϵ-transitions simplifies the construction of such an automaton: they let us "branch" between optional components (like the sign or the choice of where the digits appear) without consuming input symbols. Without ϵ-transitions, we would need to explicitly duplicate subautomata for each case, which quickly becomes cumbersome.

Use of ϵ-transitions
Keyword recognition and automata transformation

Example (Keyword recognition – compositionality)

Recognition of two keywords: web and http



Adding a new keyword can be done *compositionally*:

- build an automaton recognizing only this keyword,
- add an ϵ-transition from the general automaton's initial state to the keyword automaton's initial state,
- the unique initial state remains the one from the general automaton.

In practice, ϵ-transitions make automata transformations easier. For instance: restricting an automaton to keep only certain prefixes or suffixes of the language.

Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

Kleene closure of a language
Definition – language view

Let L be a language (not necessarily finite-state) over Σ.

Definition (Kleene closure)

The Kleene closure of L, denoted L^*, is the smallest set defined inductively by:

- $\epsilon \in L^*$, and
- if $u \in L$ and $v \in L^*$, then $u \cdot v \in L^*$.
- (equivalently: if $u \in L^*$ and $v \in L$, then $u \cdot v \in L^*$.)

Remark In other words, the Kleene closure of L is the set of all words obtained by a finite concatenation of words from L:

$$L^* = \{\epsilon\} \cup \{a_0 \cdots a_n \mid n \in \mathbb{N},\ \forall i \leq n : a_i \in L\}.$$

□

Example (Kleene closure)

- $L_1 = \{ba, cd\}$
- $L_2 = \{aa, b\}$
- $L_1^* = \{\epsilon, ba, cd, baba, bacd, cdcd, cdba, \ldots\}$
- L_2^* is the set of all words with an even number of a's and an unconstrained number of b's.

For a language consisting of words of length 1 (i.e., a subset of the alphabet), its Kleene closure is the full set of words over that alphabet.

Kleene closure of a language
Automaton view

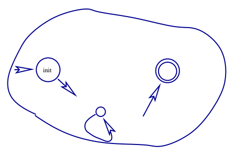
We want to show that if L is a finite-state language, then so is L^*:

$$\forall L \subseteq \Sigma^* : L \in EF \implies L^* \in EF.$$

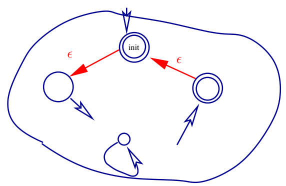
Given an automaton that recognizes L.

Can we construct an automaton that recognizes L^*?

Yes: starting from any automaton for L, we can easily obtain one for L^* using ϵ-transitions:



Automaton for L



Automaton for L^*

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NFA with ϵ-transitions

Let Σ be an alphabet where the *symbol* $\epsilon \notin \Sigma$.

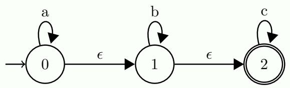
Definition (NFA with ϵ-transitions)

An *automate non-déterministe avec ϵ-transitions* (ϵ-NFA) is given by a quintuple $(Q, \Sigma, q_0, \Delta, F)$ where:

- Q is a finite set of *states*,
- Σ is the alphabet of the automaton,
- $q_0 \in Q$ is the *initial state*,
- $\Delta \subseteq Q \times (\Sigma \cup \{\epsilon\}) \times Q$ is the *transition relation*,
- $F \subseteq Q$ is the set of *accepting states*.

Example (NFA with ϵ-transitions)

Let $\Sigma = \{a, b, c\}$.



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Configuration

Let $A = (Q, \Sigma, q_0, \Delta, F)$ be an ϵ-NFA.

Definition (Configuration)

A *configuration* of the automaton A is a pair (q, u) where $q \in Q$ and $u \in \Sigma^*$.

Definition (Derivation relation (between configurations))

We define the *derivation* relation $\rightarrow_\Delta$ between configurations:

$$(q, a \cdot u) \rightarrow_\Delta (q', u')$$
iff
$$\big((q, a, q') \in \Delta \ \wedge\ u' = u\big) \quad \text{or} \quad \big(a \cdot u = u' \ \wedge\ (q, \epsilon, q') \in \Delta\big)$$

Notation

- We write $q \xrightarrow{a_1 \cdots a_n}{}^*_\Delta\, q'$ if there exist $q_1, \ldots, q_{n-1}$ such that:
$$(q, a_1, q_1) \in \Delta, (q_1, a_2, q_2) \in \Delta, \ldots, (q_{n-1}, a_n, q') \in \Delta.$$
- We write $q \longrightarrow^*_\Delta q'$ if there exist $a_1, \ldots, a_n$ such that $q \xrightarrow{a_1 \cdots a_n}{}^*_\Delta\, q'$.

Execution

Definition (Execution)

An *execution of the automaton A* is a sequence of configurations $(q_0, u_0) \cdots (q_n, u_n)$ such that
$$(q_i, u_i) \rightarrow_\Delta (q_{i+1}, u_{i+1}), \quad \text{for } i = 0, \ldots, n-1.$$

The notions of

- acceptance of a word, and
- recognized language

are defined as in the case of NFAs, *mutatis mutandis*.

Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

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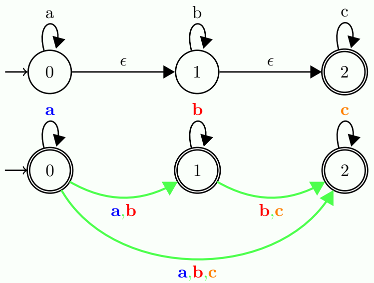
Elimination of ϵ-transitions
The idea on an example

Example (NFA with ϵ-transitions)

Let $\Sigma = \{a, b, c\}$.



Step by step, ϵ-transitions are replaced until we obtain an equivalent NFA without them.

Elimination of ϵ-transitions: translation into NFA
Definition

Let $A = (Q, \Sigma, q_0, \Delta, F)$ be an ϵ-NFA.

Definition (Elimination of ϵ-transitions)

We construct an NFA
$$\epsilon\ell(A) = (Q, \Sigma, q_0, \epsilon\ell(\Delta), \epsilon\ell(F))$$
that recognizes $L(A)$, where:

- The transition relation $\epsilon\ell(\Delta)$ is defined as follows: $(q, a, q') \in \epsilon\ell(\Delta)$ iff there exist $q_1, q_2 \in Q$ such that:
  1. $q \xrightarrow{\epsilon}{}^{*}_{\Delta} q_1$
  2. $(q_1, a, q_2) \in \Delta$
  3. $q_2 \xrightarrow{\epsilon}{}^{*}_{\Delta} q'$
- The set of accepting states $\epsilon\ell(F)$ is defined by:
$$\epsilon\ell(F) = \{q \in Q \mid \exists q' \in F : q \xrightarrow{\epsilon}{}^{*}_{\Delta} q'\}$$

Intuitively: we propagate transitions through ϵ-paths and mark states as accepting if they can reach an accepting state via ϵ-transitions

Correctness of the ϵ-elimination procedure

Let $A = (Q, \Sigma, q_0, \Delta, F)$ be an ϵ-NFA.

Theorem: Correctness of ϵ-elimination
$$L(A) = L(\epsilon\ell(A)).$$

Proof (by induction)

For all $u, u' \in \Sigma^*$ (and all $q, q' \in Q$),
$$(q, u) \longrightarrow^*_{\epsilon\ell(\Delta)} (q', u') \text{ if and only if } (q, u) \longrightarrow^*_{\Delta} (q', u').$$

- $\epsilon \in L(A)$ if and only if $\epsilon \in L(\epsilon\ell(A))$.
- Let $u \in \Sigma^*$, and assume that for all $u' \in \Sigma^*$:
$$(q, u) \longrightarrow^*_{\Delta} (q', u') \text{ iff } (q, u) \longrightarrow^*_{\epsilon\ell(\Delta)} (q', u').$$
  Let $a \in \Sigma$. We must show that for all $u' \in \Sigma^*$:
$$(q, u \cdot a) \longrightarrow^*_{\Delta} (q', u') \text{ iff } (q, u \cdot a) \longrightarrow^*_{\epsilon\ell(\Delta)} (q', u').$$

Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

ϵ-closure
Definition

The ϵ-closure of a state q is the set of all states reachable from q by following transitions labeled with ϵ.

Definition (ϵ-closure of a state)

Let $q \in Q$ be a state. We define $ECLOSE(q)$ recursively:

- *Base case:* $q \in ECLOSE(q)$
- *Induction:* If $p \in ECLOSE(q)$ and there exists a transition from p to $r \in Q$ labeled with ϵ, then $r \in ECLOSE(q)$

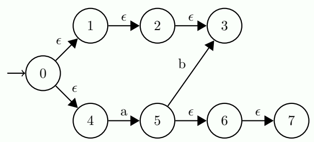
Equivalently: $ECLOSE(q) = \delta^*(q, \epsilon)$

Definition (ϵ-closure of a set of states)

For $S \subseteq Q$:
$$ECLOSE(S) = \bigcup_{q \in S} ECLOSE(q)$$

ϵ-closure
Examples

Example (ϵ-closure)



ϵ-closure of states:
- $ECLOSE(0) = \{0, 1, 2, 3, 4\}$
- $ECLOSE(3) = \{3\}$
- $ECLOSE(5) = \{5, 6, 7\}$

ϵ-closure of sets of states:
- $ECLOSE(\{0, 3\}) = \{0, 1, 2, 3, 4\}$
- $ECLOSE(\{3, 5\}) = \{3, 5, 6, 7\}$

The ϵ-closure groups together all states reachable via ϵ-transitions.

Extended Transition Relation
Definition

We define the extended transition relation $\hat{\delta}$, which allows reading both alphabet symbols and ϵ; ϵ is treated as a symbol that does not consume input.

Intuitively, $\hat{\delta}(q, w)$ is the set of states reachable by following a path whose edge labels concatenate to w (where ϵ contributes nothing to w).

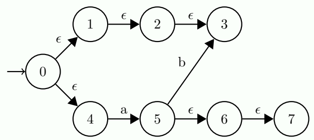
Definition (Extended transition relation)

Given $q \in Q$ and $w \in (\Sigma \cup \{\epsilon\})^*$:

- *Base case:* $\hat{\delta}(q, \epsilon) = ECLOSE(q)$
- *Induction:* For $w = x \cdot a$ with $a \in \Sigma$, $\hat{\delta}(q, x \cdot a)$ is defined as follows:
  - let $\{p_1, p_2, \ldots, p_k\} = \hat{\delta}(q, x)$,
  - let $\{r_1, r_2, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta(p_i, a)$,
  - then $\hat{\delta}(q, w) = ECLOSE(\{r_1, r_2, \ldots, r_m\})$.

Extended Transition Relation
Example

Example (Extended transition relation)



- $\hat{\delta}(0, \epsilon) = ECLOSE(0) = \{0, 1, 2, 3, 4\}$
- $\hat{\delta}(0, a) = \{5, 6, 7\}$
- $\hat{\delta}(0, b) = \emptyset$
- $\hat{\delta}(0, ab) = \{3\}$

Eliminating ϵ-Transitions and On-the-Fly Determinization
Definition of the Determinized Automaton

Let $A = (Q, \Sigma, q_0, \Delta, F)$ be an ϵ-NFA.

Definition (On-the-fly determinization and ϵ-elimination)

The *determinized* automaton of A is the DFA

$$(Q_D, \Sigma, q_D, \delta, F_D)$$

defined as follows:

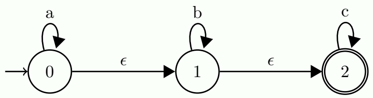
- $Q_D = \mathcal{P}(Q)$
- $q_D = ECLOSE(q_0)$
- Transition function: for any $S \in Q_D$, $a \in \Sigma$:
  - let $\{p_1, p_2, \ldots, p_k\} = S$,
  - let $\{r_1, r_2, \ldots, r_m\} = \bigcup_{i=1}^{k} \Delta(p_i, a)$,
  - then $\delta(S, a) = ECLOSE(\{r_1, r_2, \ldots, r_m\})$,
- $F_D = \{S \in \mathcal{P}(Q) \mid S \cap F \neq \emptyset\}$.

Remark Every state of the determinized automaton (reachable via δ) corresponds to an ϵ-closed set of states in the original ϵ-NFA. □

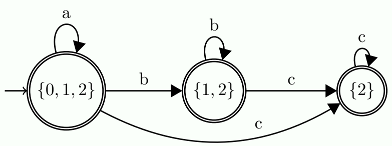
Remark In practice, we construct only the reachable ϵ-closed subsets instead of the entire powerset $\mathcal{P}(Q)$. □

Eliminating ϵ-Transitions and On-the-Fly Determinization
Translation to DFA: Example

Example (Eliminating ϵ-transitions – Translation to DFA)

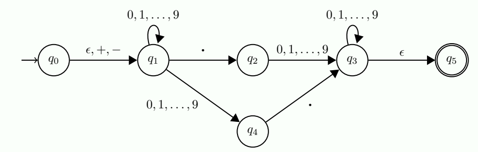


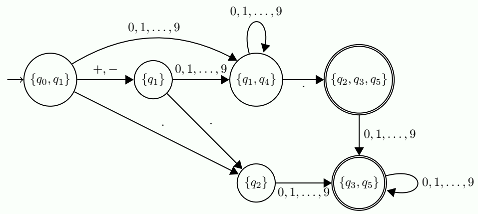
Here, elimination of ϵ-transitions is combined with determinization.
Both operations are performed *on the fly*.



Eliminating ϵ-Transitions and On-the-Fly Determinization
Translation to DFA: Another Example

Example (Eliminating ϵ-transitions – Translation to DFA)





Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

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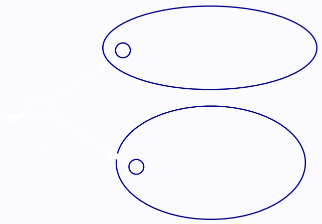
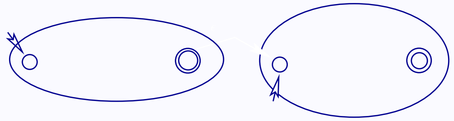
Closure of Finite-State Languages under Union and Concatenation

Union

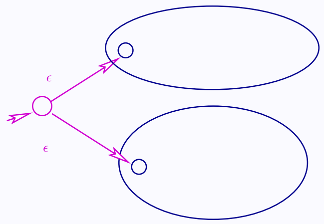


Concatenation

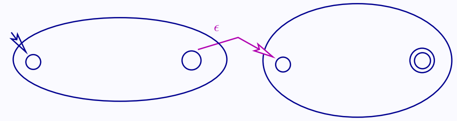


Closure of Finite-State Languages under Union and Concatenation

Union



Concatenation



Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

Reversal Operation

The reversal of a word is the word written *from right to left*.

Definition (Reversal operation – word and language)

- For $w = a_1 \cdot a_2 \cdots a_n$, the *reversal* of w is the word denoted w^R and defined by:
  $$w^R = a_n \cdot a_{n-1} \cdots a_1$$
- For $L \subseteq \Sigma^*$, the *reversal* of L is the language, denoted L^R, consisting of the reversals of words in L:
  $$L^R = \{w^R \mid w \in L\}$$

Example (Reversal)

For $L = \{001, 10, 111\}$, we have $L^R = \{100, 01, 111\}$.

Closure of Finite-State Languages under Reversal

Closure of EF under the reversal operation

- If $L \subseteq \Sigma^*$ is a finite-state language, then so is L^R.
- Hence EF is closed under the reversal operation.

Informal proof based on automata

Given a finite-state language L and its recognizing automaton A:

1. Reverse all the transitions of A.
2. Make the initial state of A the unique accepting state.
3. Create a new initial state q_0 (if the original initial state was accepting, make q_0 accepting as well).
4. Add an ϵ-transition from q_0 to each accepting state of the original automaton.

(The full proof is left as an exercise in tutorials.)

Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

Reminder: Group Morphisms

Definition (Group)

A group is a pair $(G, *)$ where G is a set and $*$ is a binary operation on G, such that for all $g_1, g_2, g_3 \in G$:

- $g_1 * g_2 \in G$,
- $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$
- there exists a neutral element e_G such that $g_1 * e_G = e_G * g_1 = g_1$
- every element has an inverse

Let $(G, \bullet)$ and $(G', *)$ be two groups whose neutral elements are e_G and $e_{G'}$, respectively.

Definition (Morphism)

A function $f : G \to G'$ is a group morphism if

$$\forall x, y \in G : f(x \bullet y) = f(x) * f(y)$$

Example (Morphism)

The function $f : (\mathbb{Z}, +) \to (\mathbb{R}^+, \times)$ defined by $f(n) = 2^n$ is a group morphism.

In what follows, for each alphabet Σ, we consider the group $(\Sigma^*, \cdot)$ where $\cdot$ is the concatenation of words over Σ^*, and morphisms are used to translate words over one alphabet into words over another.

Morphisms on Words

Let Σ and Σ' be two alphabets.

Definition (Word Morphism)

A function $h : \Sigma \to \Sigma'^*$ induces a morphism $\hat{h} : \Sigma^* \to \Sigma'^*$ defined by:

- $\hat{h}(\epsilon) = \epsilon$, and
- $\hat{h}(u \cdot a) = \hat{h}(u) \cdot h(a)$.

Remark We can prove that $\hat{h}$ is a morphism by showing that

$$\forall x, y \in \Sigma^* : \hat{h}(x \cdot y) = \hat{h}(x) \cdot \hat{h}(y),$$

using induction on y (or equivalently on $|y|$). □

Example (Word Morphism)

Consider $\Sigma = \{a, b\}$, $\Sigma' = \{0, 1\}$ and the function $h : \Sigma \to \Sigma'^*$ defined by $h(a) = 0$ and $h(b) = 1 \cdot 1$.
This function h induces a morphism $\hat{h} : \Sigma^* \to \Sigma'^*$.
For instance,

$$\hat{h}(b \cdot a \cdot a) = 1 \cdot 1 \cdot 0 \cdot 0 = \hat{h}(b) \cdot \hat{h}(a \cdot a).$$

From now on, we write h instead of $\hat{h}$.

Closure of Finite-State Languages under Morphisms

Theorem: Closure of EF under Morphisms

If $L \subseteq \Sigma^*$ is a finite-state language, then so is its image under a morphism h, denoted $h(L)$ and defined by

$$h(L) = \{h(u) \mid u \in L\}.$$

Thus EF is closed under morphisms.

Proof

Based on automata. Left as an exercise.

Example (Closure of EF under Morphisms)

Consider $\Sigma = \{a, b\}$, $\Sigma' = \{0, 1\}$ and the morphism $\hat{h}$ (denoted simply h below) induced by the function $h : \Sigma \to \Sigma'^*$ defined by $h(a) = 0$ and $h(b) = 1 \cdot 1$.

- The language $L_1 \subseteq \Sigma^*$ consisting of all words with an odd number of a's is a finite-state language.
- The language $h(L_1) \subseteq \Sigma'^*$ consisting of all words with an odd number of 0's and an even number of 1's is also a finite-state language.

Intuition

Applying a morphism corresponds to *relabeling or expanding transitions* in the automaton of L. Each letter $a \in \Sigma$ is replaced by the automaton for $h(a)$, so the structure remains finite-state. This guarantees that $h(L)$ is recognized by a finite automaton.

Closure of EF under Inverse Morphism

Let h be a morphism and h^{-1} its inverse mapping (preimage under h).

Theorem: Closure of EF under Inverse Morphism

If $L \subseteq \Sigma'^*$ is a finite-state language, then so is its preimage under h, defined by

$$h^{-1}(L) = \{u \in \Sigma^* \mid h(u) \in L\}.$$

Thus EF is closed under inverse morphisms.

Proof

Based on automata. Left as an exercise.

Example (Closure of EF under Inverse Morphism)

Let $\Sigma = \{a, b, c, d\}$, $\Sigma' = \{0, 1, 2\}$, and consider the morphism $h : \Sigma \to \Sigma'^*$ defined by

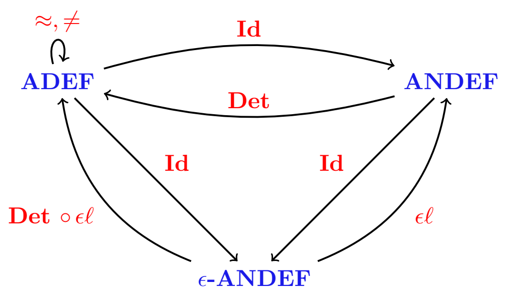
$$h(a) = 0, \quad h(b) = 1, \quad h(c) = \epsilon, \quad h(d) = 2.$$

- The language $L_1 \subseteq \Sigma'^*$ consisting of all words with an even number of 0's and no 2's is finite-state.
- Its preimage $h^{-1}(L_1) \subseteq \Sigma^*$ consists of all words with:
  - an even number of a's,
  - no d's,
  - arbitrary occurrences of b and c.

  This is also finite-state.

Outline Chap. 8 - Non-deterministic Finite-state Automata w/ ϵ-transitions

Summary 1: Transformations Between Automata



Summary 2: Closure of the Class of Finite-State Languages, Decision Problems and Procedures

Closure Properties

Finite-state languages are closed under the following operations:

1. union, intersection,
2. complement,
3. concatenation,
4. Kleene star (closure),
5. reversal,
6. morphism, inverse morphism.

Each closure property was shown via a corresponding automaton construction.

Decision Problems and Procedures

The following decision problems are decidable:

1. state reachability,
2. co-reachability (∃ a path to a final state),
3. emptiness,
4. infiniteness,
5. language inclusion,
6. language equivalence.

For each problem, we provided a concrete decision procedure.