

Programming Language Semantics and Compiler Design

(Sémantique des Langages de Programmation et Compilation) Axiomatic Semantics - Hoare Logic

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Master of Sciences in Informatics at Grenoble (MoSIG) Master 1 info

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Outline - Axiomatic Semantics - Hoare Logic

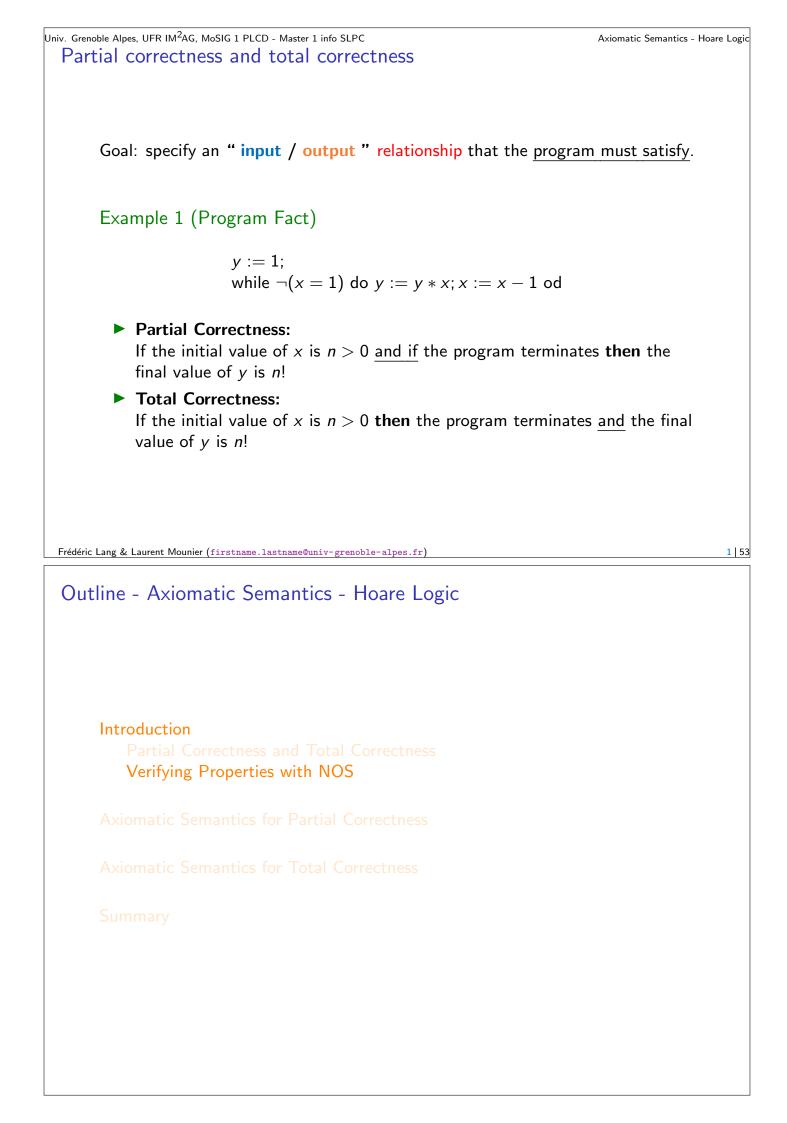
Introduction

Axiomatic Semantics for Partial Correctness

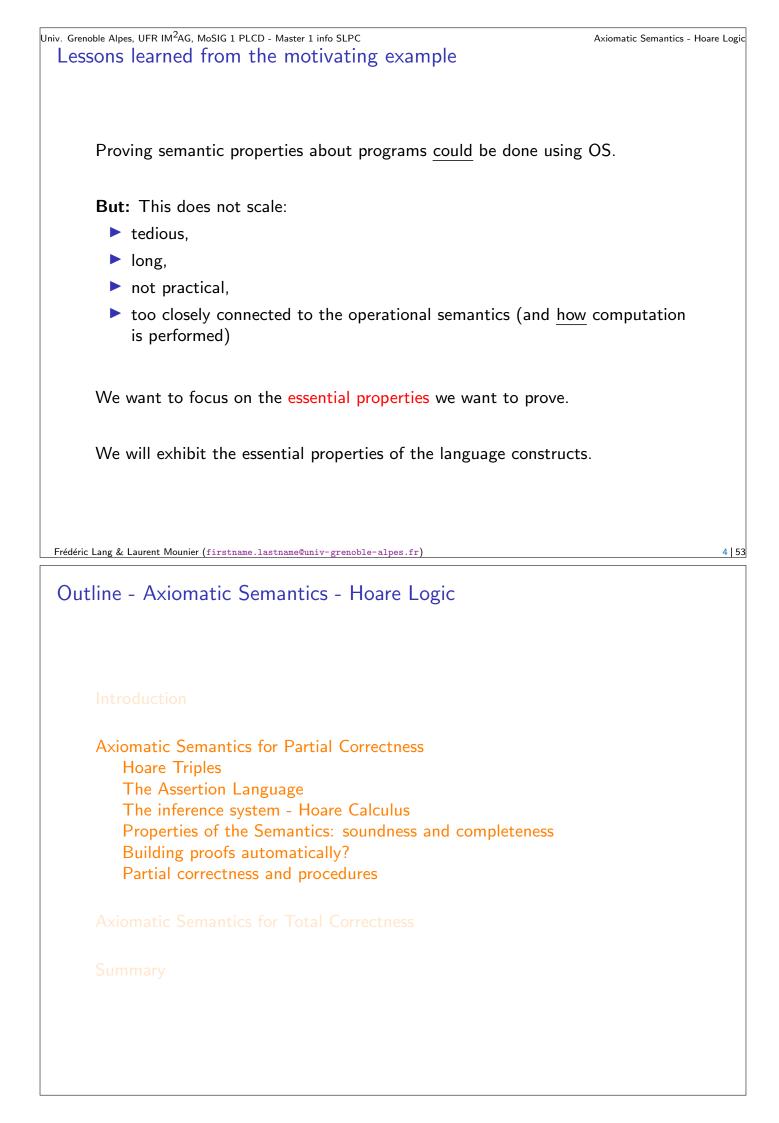
Axiomatic Semantics for Total Correctness

Summary

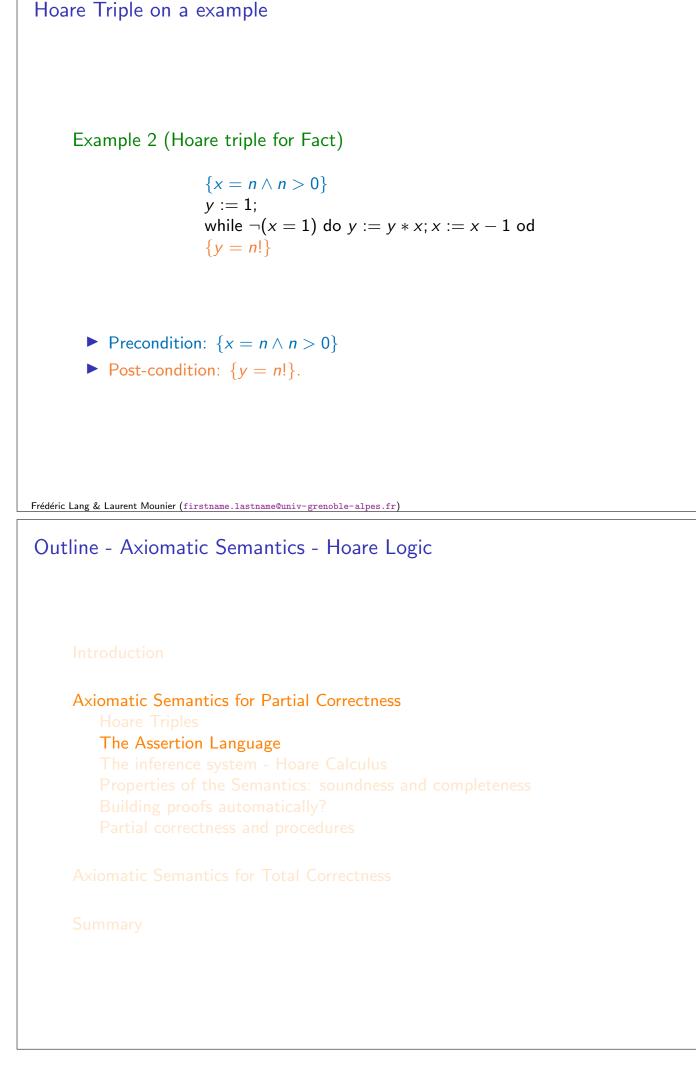
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	Introduction Partial Correctness and Total Correctness Verifying Properties with NOS



Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic Verifying semantic properties - a motivating example How can we prove the (partial) correctness of Fact using NOS? Fact: v := 1: while $\neg(x = 1)$ do y := y * x; x := x - 1 od Formalization: Let *n* be the initial value of xn > 0 and $\sigma_0(x) = n$ and (Fact, σ_0) $\rightarrow \sigma'$ implies $\sigma'(y) = n!$ Stage 0 Correctness of the initialization. Stage 1 Correctness of the loop body. Stage 2 Correctness of the loop. Stage 3 Correctness of the program. \hookrightarrow Study of the derivation tree. Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr) 2 | 53 Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic Verifying semantic properties - a motivating example (continued) Correctness of Fact Use of a loop invariant $I(\sigma) \stackrel{\text{def}}{=} \sigma(y) = \frac{n!}{\sigma(x)!}$ Stage 0 The initialization y := 1 satisfies for all σ, σ' : if n > 0 and $\sigma(x) = n$ and $(y := 1, \sigma) \to \sigma'$ then $I(\sigma')$ and $\sigma'(x) > 0$ Stage 1 The loop body satisfies for all σ, σ' : if $I(\sigma)$ and $\sigma(x) > 0$ and $(y := y * x; x := x - 1, \sigma) \rightarrow \sigma'$ then $I(\sigma')$ Stage 2 The loop satisfies for all σ, σ' : if $I(\sigma)$ and $\sigma(x) > 0$ and (while $\neg(x = 1)$ do \ldots od, $\sigma) \rightarrow \sigma'$ then $I(\sigma')$ and $\sigma'(x) = 1$ Note that $I(\sigma')$ and $\sigma'(x) = 1$ implies $\sigma'(y) = n!$ Stage 3 Partial correctness of the program: if n > 0 and $\sigma(x) = n$ and (Fact, σ) $\rightarrow \sigma'$ then $\sigma'(y) = n!$ Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr) 3 | 53

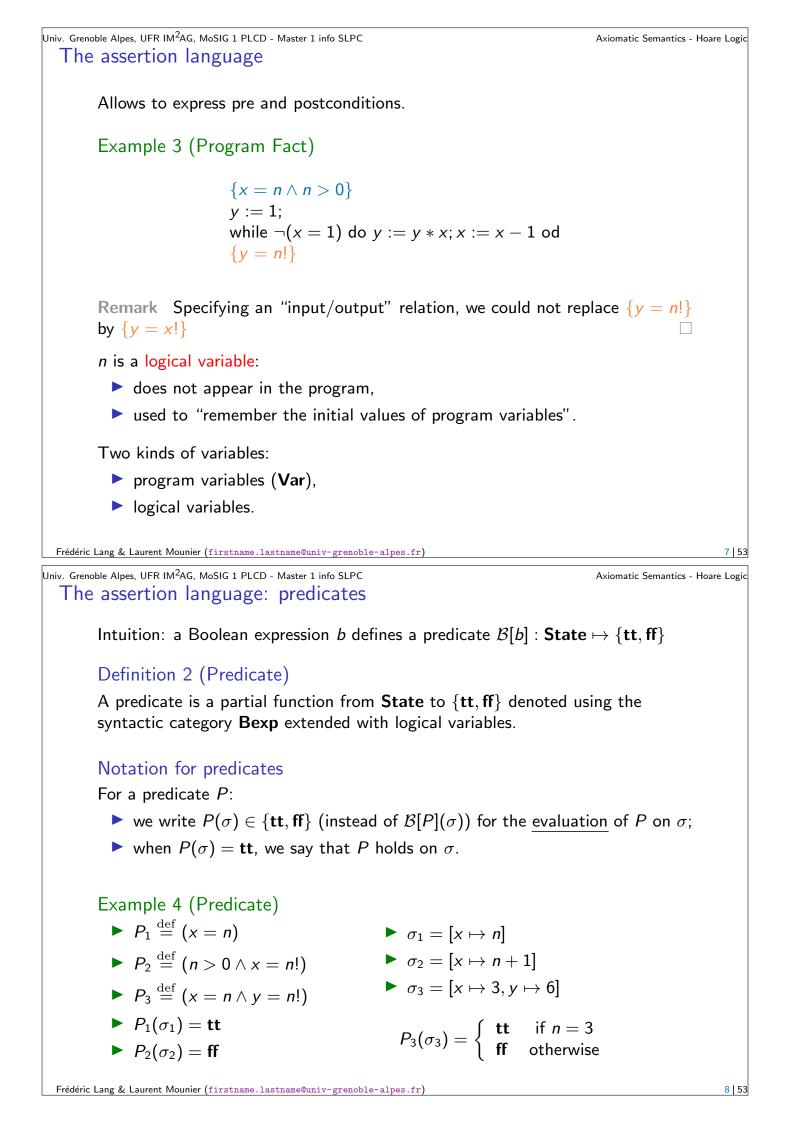


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Hoare Triples		
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Idea: specify propert	ties of programs as relations l	
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Idea: specify propertion inputs and propertien Definition 1 (Hoard S a statement	s of its outputs via <u>Hoare trip</u> e Triple) { <i>P</i> } <i>S</i> { <i>Q</i> }	
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 Idea: specify propertion inputs and properties Definition 1 (Hoard S a statement P an assertion, Q an assertion, 	s of its outputs via <u>Hoare trip</u> e Triple) { <i>P</i> } <i>S</i> { <i>Q</i> } called precondition	
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Idea: specify propertion inputs and propertien Definition 1 (Hoard S a statement P an assertion, Q an assertion, Meaning: <u>if</u> P <u>and</u> <u>ther</u> If we can prove this,	s of its outputs via <u>Hoare trip</u> e Triple) $\{P\} S \{Q\}$ called precondition called postcondition holds in the initial state (bef the execution of <i>S</i> on that state <i>Q</i> will hold in the state in v we say that $\{P\} S \{Q\}$ hold	oles. Fore executing <i>S</i>) tate <u>terminates</u> , which <i>S</i> terminates.



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Reminder: The Boolean domain $\{\mathbf{tt}, \mathbf{ff}\}$ is endowed with usual Boolean operators, noted \land , \lor , \neg , and \Longrightarrow , and their usual semantics.

Notations (Boolean operators on predicates)

For all predicates P_0, P_1, P_2 and all state $\sigma \in$ **State**:

- $P_1 \wedge P_2$ denotes the function defined by $(P_1 \wedge P_2)(\sigma) \stackrel{\text{def}}{=} P_1(\sigma) \wedge P_2(\sigma)$,
- $P_1 \vee P_2$ denotes the function defined by $(P_1 \vee P_2)(\sigma) \stackrel{\text{def}}{=} P_1(\sigma) \vee P_2(\sigma)$,
- ▶ $\neg P_0$ denotes the function defined by $(\neg P_0)(\sigma) \stackrel{\text{def}}{=} \neg (P_0(\sigma))$,
- $P_1 \implies P_2 \text{ denotes the function defined by}$ $(P_1 \implies P_2)(\sigma) \stackrel{\text{def}}{=} P_1(\sigma) \implies P_2(\sigma),$

Example 5 (Predicate) Recall that $P_1 \stackrel{\text{def}}{=} (x = n)$ and $P_2 \stackrel{\text{def}}{=} (x = n!)$:

$$(P_1 \wedge P_2)([x \mapsto 2]) = \left\{ egin{array}{cc} {f tt} & {
m if} \ n=2 \ {f ff} & {
m otherwise} \end{array}
ight.$$

If n is unknown and $P_1 \wedge P_2$ is assumed to hold, then it implies that n = 2.

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Definition 3 (Logical equivalence)

Two predicates P_1 and P_2 are logically equivalent iff for all σ such that $vars(P_1) \subseteq dom(\sigma)$ and $vars(P_2) \subseteq dom(\sigma)$, $P_1(\sigma) = P_2(\sigma)$.

Remark If P_1 and P_2 are logically equivalent, then they may be freely interchanged (e.g., predicate simplification).

Definition 4 (Substitution)

For $x \in Var$ and $a \in Aexp$, P[a/x] is a predicate obtained by replacing (syntactically) each occurrence of x by a in P.

Example 6 (Substitution)

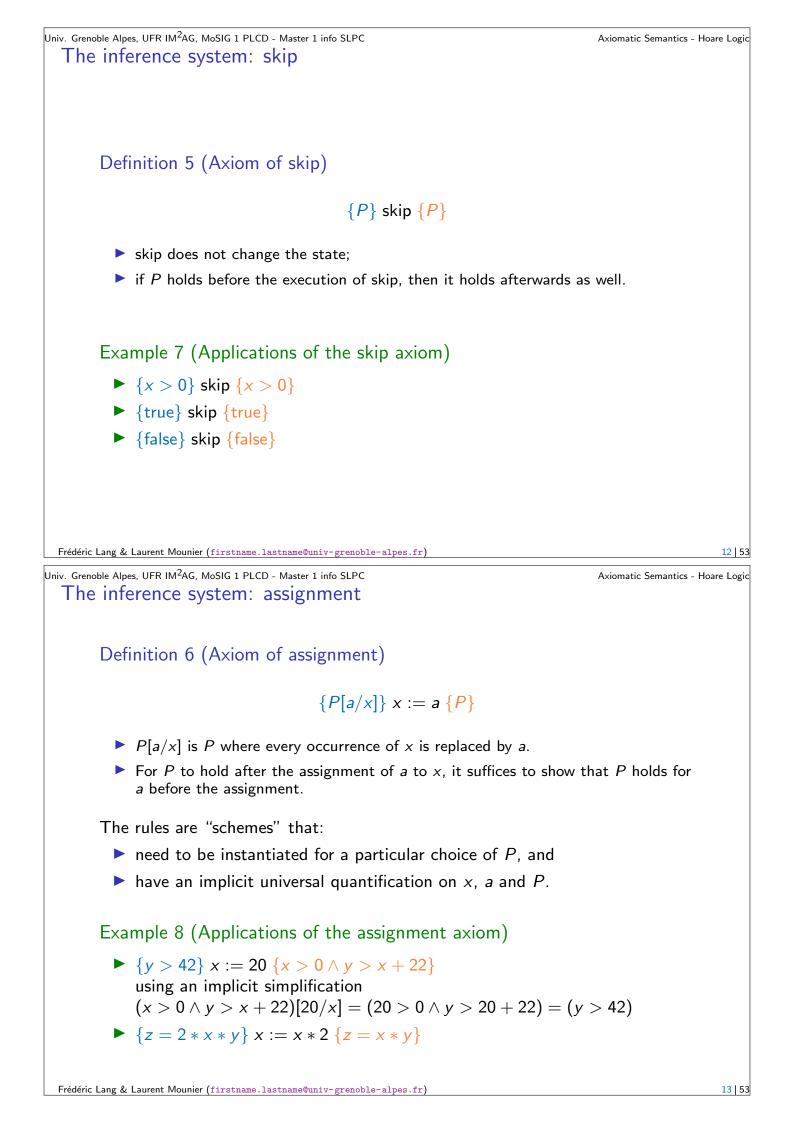
Recall that $P_1 \stackrel{\text{def}}{=} (x = n)$:

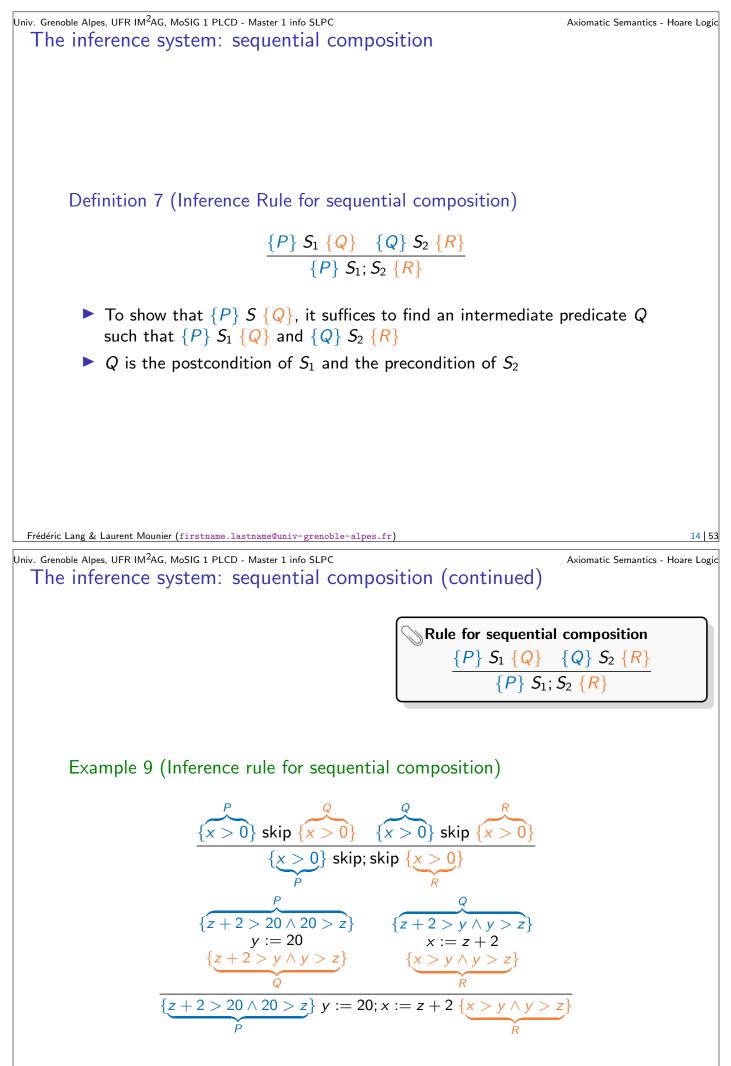
 $\blacktriangleright P_1[y+2/x] \stackrel{\text{def}}{=} (y+2=n)$

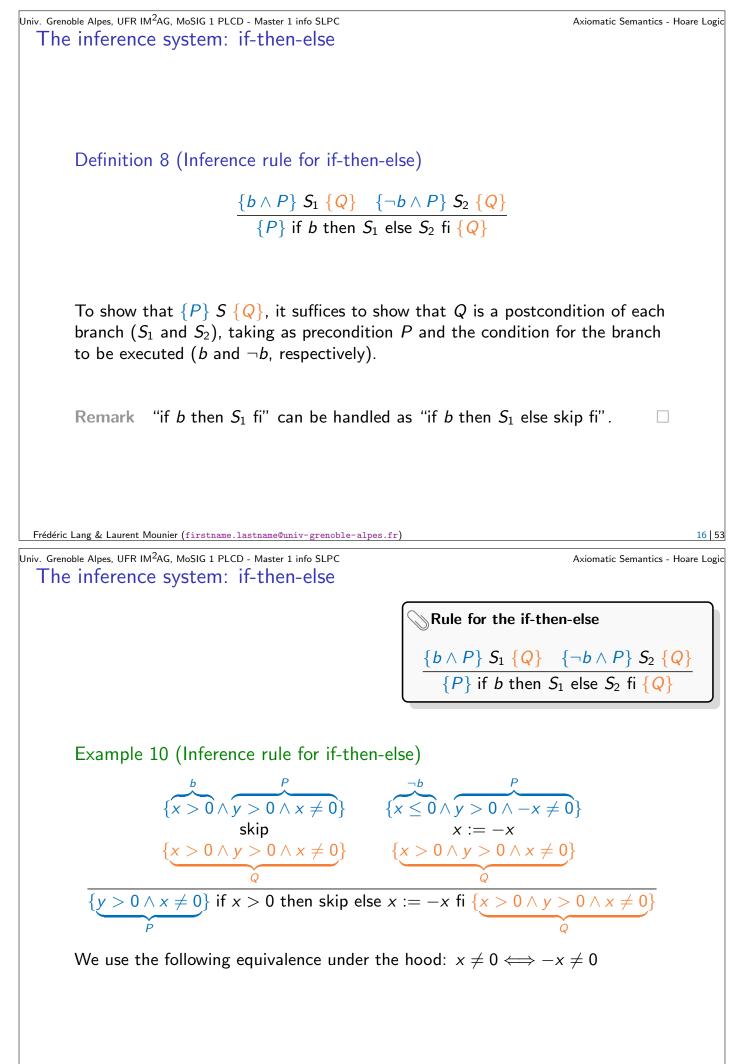
Remark Logical equivalence is closed under substitution: If P_1 and P_2 are logically equivalent, then for all $x \in$ Var, $a \in$ Aexp, $P_1[a/x]$ and $P_2[a/x]$ are logically equivalent.

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line - Axiomatic Semantics - Hoare Logic	
Axiomatic Semantics for Partial Correctness	
The inference system - Hoare Calculus	
uble Alpes, UFR IM ² AG, MoSIG 1 PLCD - Master 1 info SLPC	Axiomatic Semantics -
bble Alpes, UFR IM ² AG, MoSIG 1 PLCD - Master 1 info SLPC inference system - Hoare Calculus Recall logical derivation with rules:	Axiomatic Semantics -
inference system - Hoare Calculus	Axiomatic Semantics -
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inference system - Hoare Calculus Recall logical derivation with rules: <u> <u> Premisse1</u> Premissen <u> Conclusion</u> ► <u>forward</u> interpretation/reading: if we have proved Prem</u>	
 inference system - Hoare Calculus Recall logical derivation with rules: <u>Premisse1 Premissen</u> <u>Conclusion</u> <u>forward interpretation/reading: if we have proved Premissen</u>, then we have proved Conclusion; 	<i>isse</i> 1 and and
inference system - Hoare Calculus Recall logical derivation with rules: <u>Premisse₁</u> Premisse _n <u>Conclusion</u> ► <u>forward</u> interpretation/reading: if we have proved Prem	<i>isse</i> 1 and and
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 inference system - Hoare Calculus Recall logical derivation with rules: <u>Premisse1 Premissen</u> <u>forward</u> interpretation/reading: if we have proved Premissen, then we have proved Conclusion; <u>backward</u> interpretation/reading: to prove Conclusion, is Premisse1 and and Premissen. Partial correctness assertions will be specified by an inference 	<i>isse</i> ₁ and and t suffices to prove
 inference system - Hoare Calculus Recall logical derivation with rules: <u>Premisse1 Premissen</u> <u>Conclusion</u> forward interpretation/reading: if we have proved Premissen, then we have proved Conclusion; <u>backward</u> interpretation/reading: to prove Conclusion, is <u>Premisse1</u> and and Premissen. Partial correctness assertions will be specified by an inference and rules) that will allow us to write inference trees. 	<i>isse</i> ¹ and and t suffices to prove e system (axioms
 inference system - Hoare Calculus Recall logical derivation with rules: <u>Premisse1 Premissen</u> <u>Conclusion</u> forward interpretation/reading: if we have proved Premissen, then we have proved Conclusion; <u>backward</u> interpretation/reading: to prove Conclusion, is <u>Premisse1</u> and and Premissen. Partial correctness assertions will be specified by an inference and rules) that will allow us to write inference trees. 	<i>isse</i> ₁ and and t suffices to prove e system (axioms
 inference system - Hoare Calculus Recall logical derivation with rules: <u>Premisse₁ Premissen</u> <u>Conclusion</u> <u>forward</u> interpretation/reading: if we have proved Prem <u>Premissen</u>, then we have proved Conclusion; <u>backward</u> interpretation/reading: to prove Conclusion, if <u>Premisse1</u> and and Premissen. Partial correctness assertions will be specified by an inference and rules) that will allow us to write inference trees. Intuitively, an inference tree says how properties "propagate" 	<i>isse</i> ₁ and and t suffices to prove e system (axioms







Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic The inference system: while loop Definition 9 (Inference rule for while loop) $\frac{\{b \land P\} S \{P\}}{\{P\} \text{ while } b \text{ do } S \text{ od } \{\neg b \land P\}}$ P is a loop invariant, which must hold before and after each execution of the loop-body. Example 11 (Inference rule for while loop) $\begin{cases} x > 0 \land I \\ x := x - 1 \\ \{x \ge 0 \land y + z = z * (x_0 - x) \land z = z_0 \} \\ x \ge 0 \land y + z = z * (x_0 - x) \land z = z_0 \end{cases}$ $\begin{cases} x \ge 0 \land y + z = z * (x_0 - x) \land z = z_0 \\ \{I \} \end{cases}$ $\frac{\{x \ge 0 \land y + z = z * (x_0 - x) \land z = z_0 \} \\ \{I \} \end{cases}$ $\frac{\{x \ge 0 \land I \} x := x - 1; y := y + z \{I \} \\ \hline \{I \} \text{ while } x > 0 \text{ do } x := x - 1; y := y + z \text{ od } \{x \le 0 \land I \} \end{cases}$ $\{x \geq 0 \land y + z = z * (x_0 - x) \land z = z_0\}$ where $I \stackrel{\text{def}}{=} x \ge 0 \land y = z * (x_0 - x) \land z = z_0$ 18 53 Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr) Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic Inference system We now have a rule for each statement of While. Is it sufficient? Exercise 1 Prove $\{x > 0\} x := x + 1 \{x > 0\}$. This is not possible so far... Using the axiom of assignment, we can only build either proof tree: $\overline{\{x > 0\} \ x := x + 1 \ \{x > 1\}}$ or $\overline{\{x > 0\} \ x := x + 1 \ \{x > 0\}}$ Can we replace the postcondition x > 1 of the first Hoare triple or the precondition $x \ge 0$ of the second Hoare triple by x > 0? Yes, because $x > 1 \implies x > 0$ and $x > 0 \implies x > 0$. This must be allowed by a rule: the rule of consequence.

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If $P \implies Q$, we say that: \triangleright P is stronger than Q (i.e., P sets more constraints on the state than Q) Q is weaker than P. Reminder: $P \implies Q$ means "if P holds then Q holds". If P does not hold, then it sets no constraint on the state. It is equivalent to $\neg P \lor Q$. For all *P*, we have both: $\triangleright P \implies true$ true is the weakest predicate, setting no constraint at all \blacktriangleright false $\implies P$ false is the strongest predicate, setting a constraint that cannot be fulfilled Example 12 x > 0 is stronger than $x \ge 0$, because if if x > 0 then necessarily $x \ge 0$. Neither of x > 0 and x < 3 is stronger or weaker than the other. Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr) 20 | 53 Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic The inference system: consequence

The following rule allows the precondition to be weakened and/or the postcondition to be strengthened.

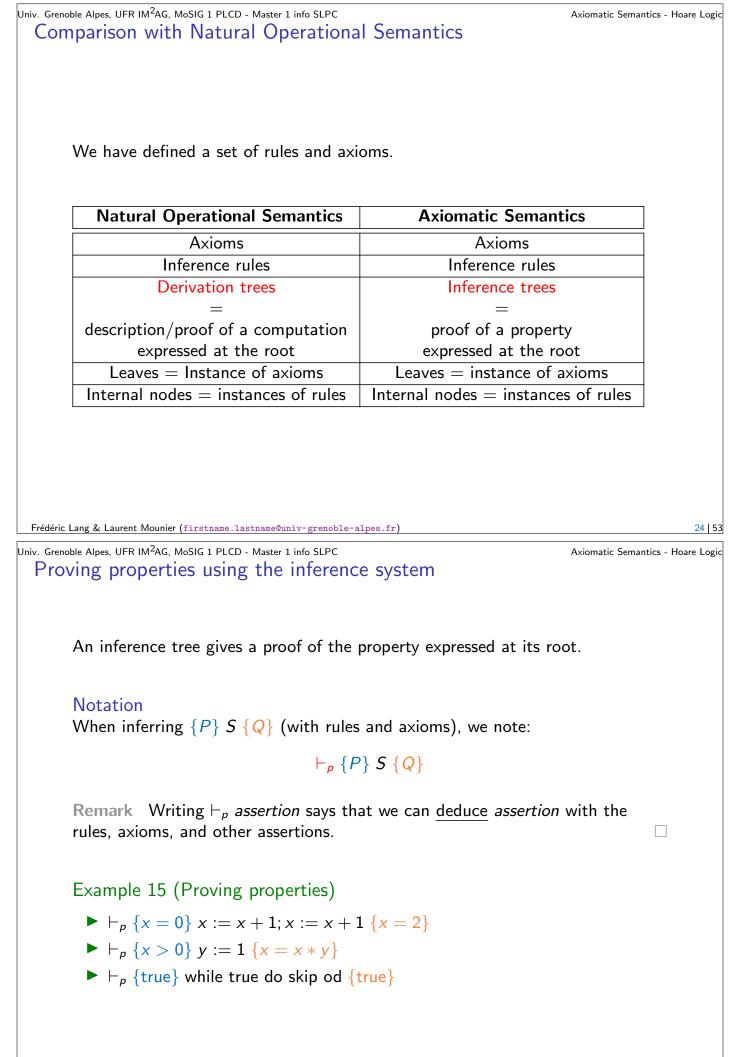
Definition 10 (Inference rule for consequence) If $P \implies P'$ and $Q' \implies Q$, then:

$$\frac{\{P'\} S \{Q'\}}{\{P\} S \{Q\}}$$

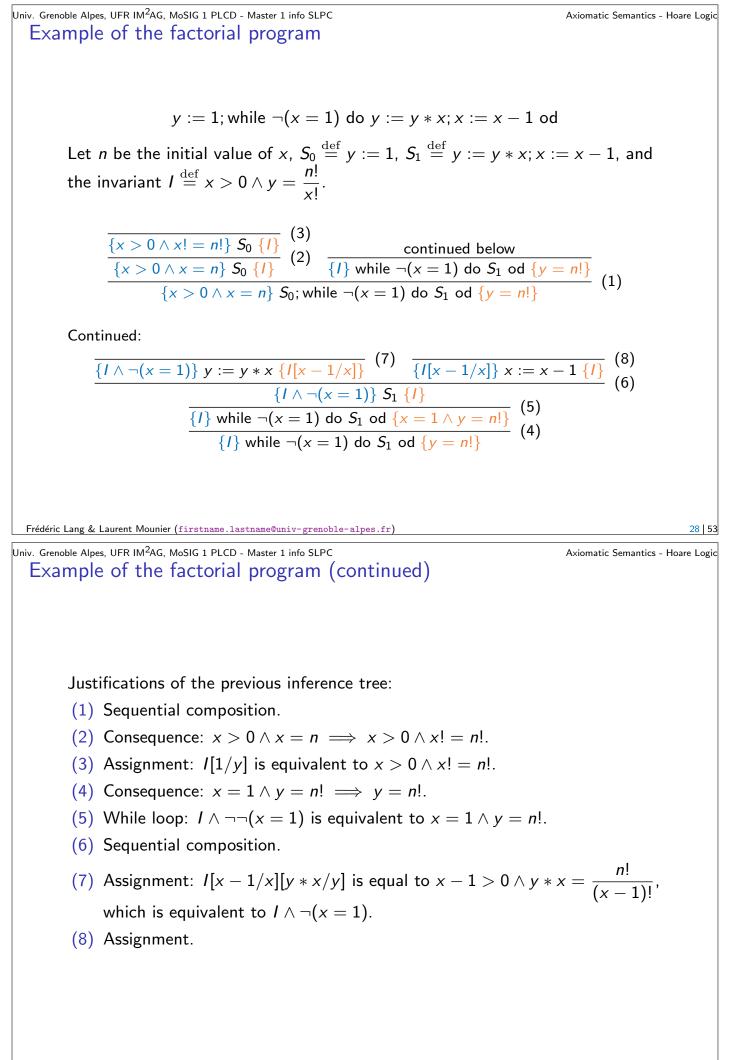
Example 13 (Inference rule for consequence)

$$\frac{\{x \ge 0\} \ y := x + 1 \ \{y > 0\}}{\{x > 42\} \ y := x + 1 \ \{y \ne 0\}}$$

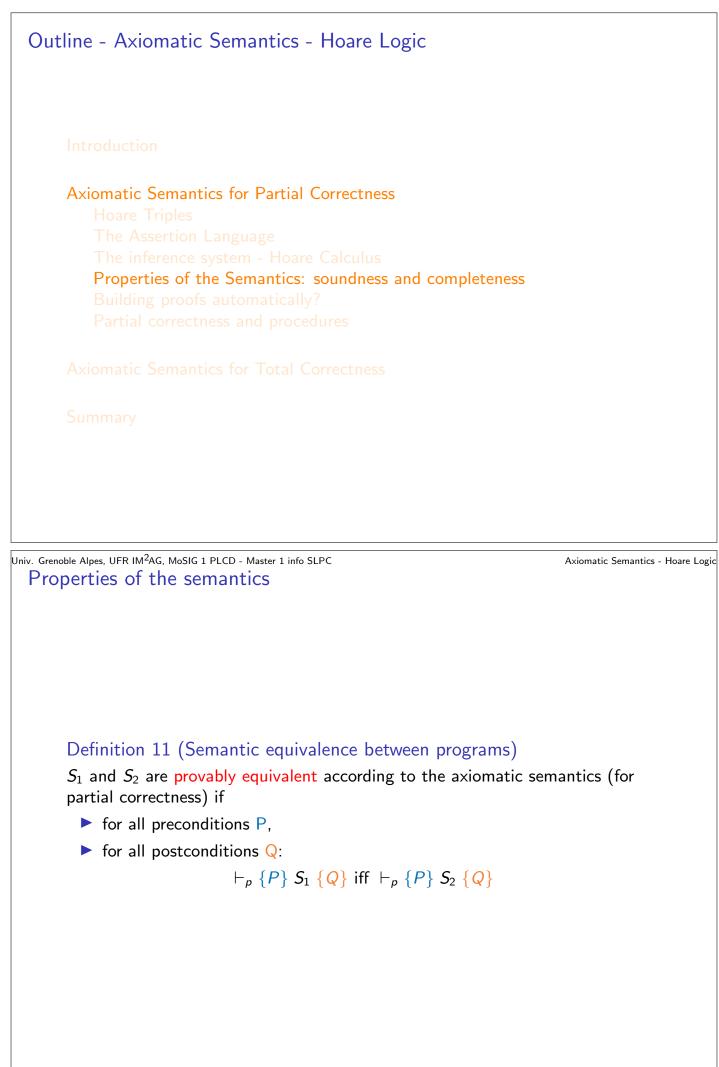
Building	_	PLCD - Master 1 info SLPC g Hoare logic	Axiomatic Semantics - Hoare Logic
	In the NOS of $(S, \sigma) ightarrow \sigma'$ is applies, determination conditions. Building proof	While, the derivation tree built to achieve a g unique, because at each step at most one infer nined by the program syntax and the value of E s using Hoare logic is less deterministic, as ther of the same Hoare triple. For instance, the rule	rence rule Boolean re may exist
	consequence is	not guided by the program syntax.	
	nple 14		
Here		s of the same Hoare triple $\{x > 0\} x := x + 1$ = $x + 1 \{x > 1\}$ = $x + 1 \{x > 0\}$ = $x + 1 \{x > 0\}$ $\frac{\{x \ge 0\} x := x + 1 \{x > 0\}}{\{x > 0\} x := x + 1 \{x > 0\}}$	
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	UFR IM ² AG, MoSIG 1 plete inferei	PLCD - Master 1 info SLPC ICE System	Axiomatic Semantics - Hoare Logic
	Rule name	rule	
	Skip	{ <i>P</i> } skip { <i>P</i> }	
	Assignment	$\{P[a/x]\} x := a \{P\}$	
	Assignment Sequential		
		$\{P[a/x]\} := a \{P\}$	
	Sequential	$\{P[a/x]\} x := a \{P\}$ $\frac{\{P\} S_1 \{Q\} \{Q\} S_2 \{R\}}{\{P\} S_1; S_2 \{R\}}$	
	Sequential	$\{P[a/x]\} x := a \{P\}$ $\frac{\{P\} S_1 \{Q\} \{Q\} S_2 \{R\}}{\{P\} S_1; S_2 \{R\}}$ $\frac{\{b \land P\} S_1 \{Q\} \{\neg b \land P\} S_2 \{Q\}}{\{P\} \text{ if } b \text{ then } S_1 \text{ else } S_2 \text{ fi } \{Q\}}$ $\{b \land P\} S \{P\}$	

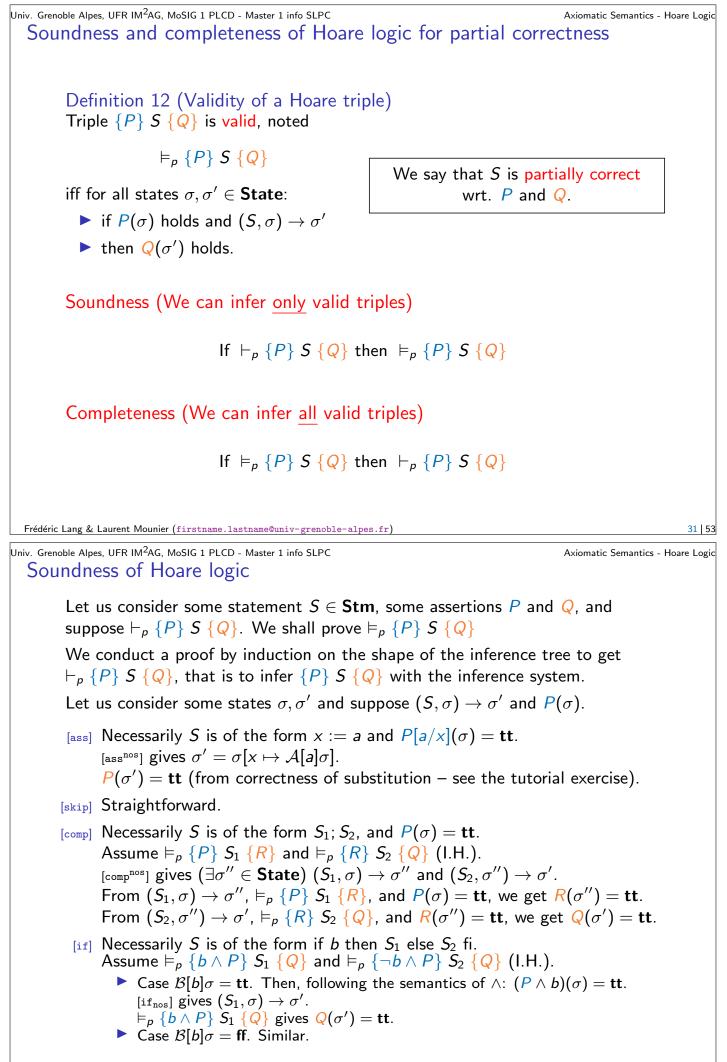


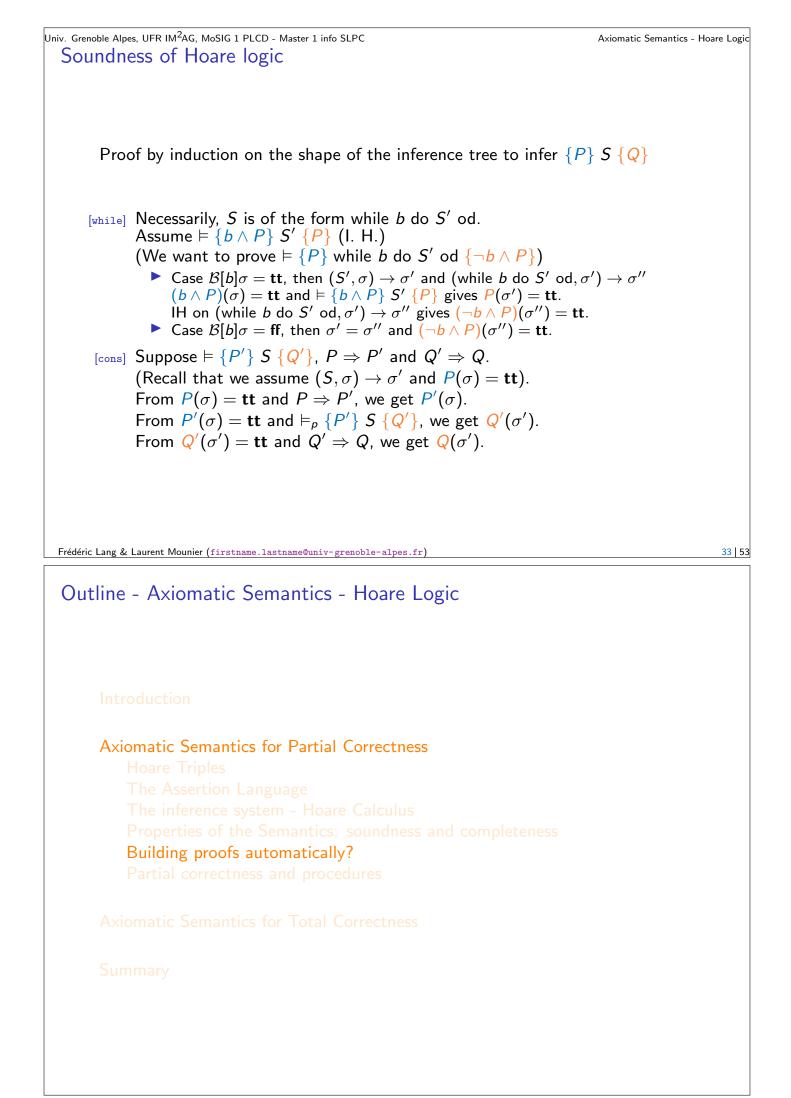
Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic Exercises: a solution Proof of $\vdash_{p} \{x = 0\} x := x + 1; x := x + 1 \{x = 2\}$ $\frac{\{x=0\} \ x := x+1 \ \{x=1\}}{\{x=0\} \ x := x+1 \ \{x=1\}} \begin{bmatrix} ass \\ x = 1 \end{bmatrix} x := x+1 \ \{x=2\}} \begin{bmatrix} ass \\ comp \end{bmatrix}}{[comp]}$ because (x = 1)[x + 1/x] = (x + 1 = 1) = (x = 0)(x = 2)[x + 1/x] = (x + 1 = 2) = (x = 1)Proof of $\vdash_p \{x > 0\} y := 1 \{x = x * y\}$ $\frac{\overline{\{\text{true}\} y := 1 \{x = x * y\}}}{\{x > 0\} y := 1 \{x = x * y\}} \text{ [conseq]}$ because (x = x * y)[1/y] = (x = x) = true $x > 0 \implies true$ 26 | 53 Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr) Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic Exercises: a solution Proof of \vdash_p {true} while true do skip od {true} $\frac{ \{ true \} skip \{ true \} }{ \{ true \} while true do skip od \{ false \} } { true \} while true do skip od \{ true \} } [conseq]$ \neg true \land true = false because false \implies true **Remark** If $\{P\}$ S $\{Q\}$ and S is started in a state satisfying P, we cannot claim that S will terminate in a state satisfying Q



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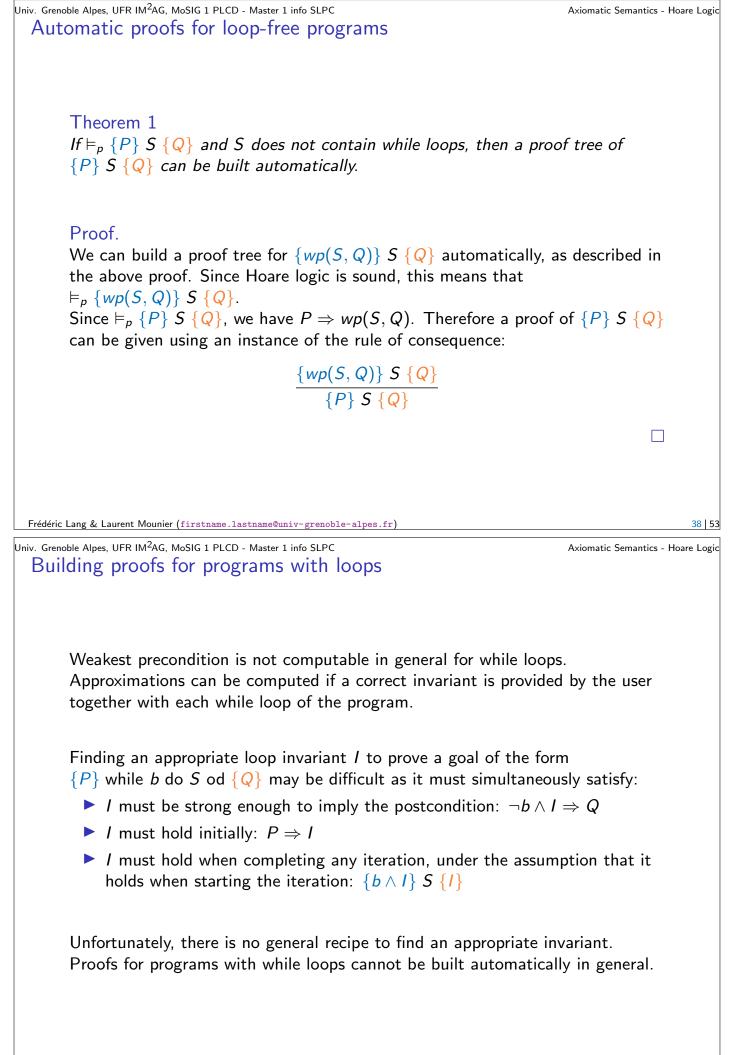




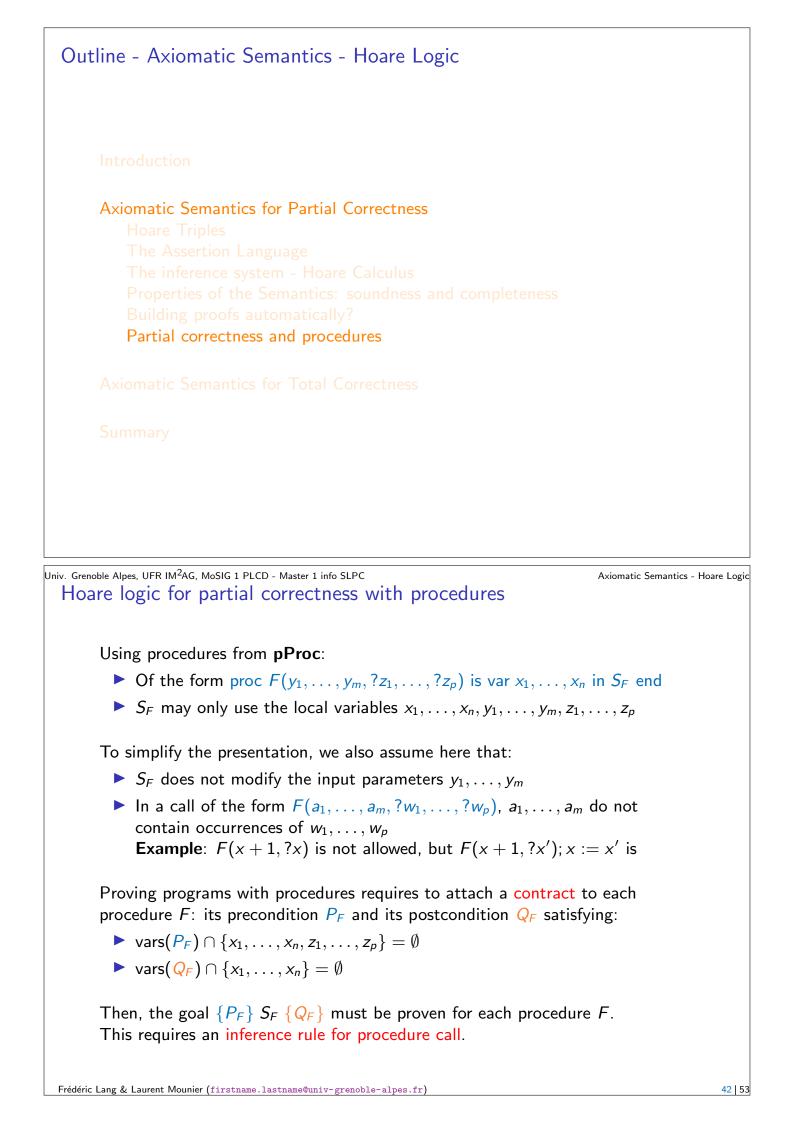
Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic Building proofs using the weakest-precondition calculus For a statement S without while loop, if $\vDash_p \{P\} S \{Q\}$, then a proof can be built systematically using a calculus called the weakest-precondition calculus (wp). Given the postcondition Q and the loop-free statement S, this calculus computes the weakest precondition wp(S, Q) such that $\models_p \{wp(S, Q)\} S \{Q\}$. This precondition is called weakest precondition because $\models_p \{P\} S \{Q\}$ holds if and only if $P \Rightarrow wp(S, Q)$. Definition 13 (Weakest precondition calculus for loop-free statements) $\begin{array}{rcl} wp(\mathsf{skip},Q) &=& Q\\ wp(x:=a,Q) &=& Q[a/x]\\ wp(\mathsf{if}\ b\ \mathsf{then}\ S_1\ \mathsf{else}\ S_2\ \mathsf{fi},Q) &=& (b\Rightarrow wp(S_1,Q)) \wedge (\neg b\Rightarrow wp(S_2,Q))\\ wp(S_1;S_2,Q) &=& wp(S_1,wp(S_2,Q)) \end{array}$ We shall prove that $\vdash_p \{wp(S, Q)\} \in \{Q\}$, i.e., wp(S, Q) is indeed a valid precondition. This proof is constructive, in the sense that we can deduce from it an algorithm that actually builds a proof tree. The proof that it is the weakest precondition is out of the scope of this lecture. Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr) 34 | 53 Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic From wp to proof trees: skip and assignment Let S be any loop-free statement. We show by induction on S that $(\forall Q) \vdash_{p} \{wp(S, Q)\} S \{Q\}$ Skip wp(skip, Q) = QThe proof tree is a valid instance of the axiom of skip: $\{Q\}$ skip $\{Q\}$ Assignment wp(x := a, Q) = Q[a/x]The proof tree is a valid instance of the axiom of assignment: $\{Q[a/x]\} x := a \{Q\}$

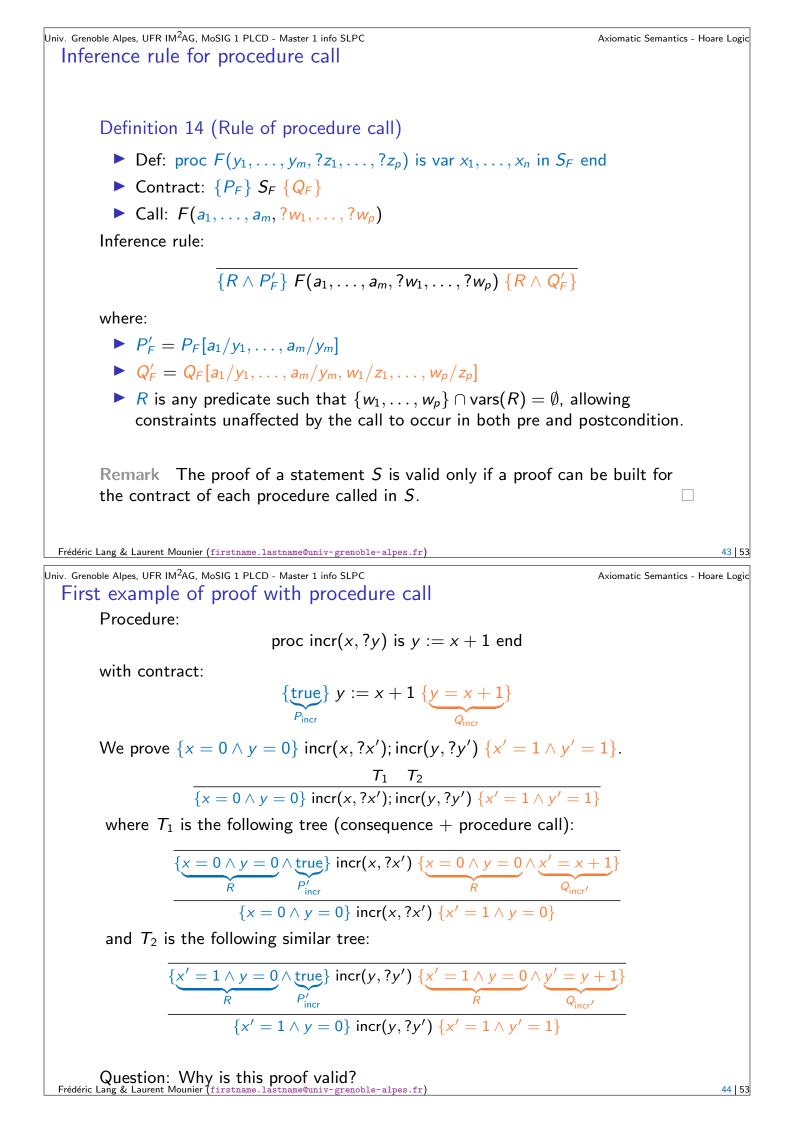
Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic From wp to proof trees: sequential composition $wp(S_1; S_2, Q) = wp(S_1, wp(S_2, Q))$ Let $R = wp(S_2, Q)$ and $P = wp(S_1, R)$. The proof tree is an instance of the inference rule for sequential composition: $\frac{\{P\} S_1 \{R\} \{R\} S_2 \{Q\}}{\{P\} S_1; S_2 \{Q\}}$ The premises have valid proof trees from the induction hypothesis: $\vdash_{p} \{\underbrace{wp(S_{1},R)}_{p}\} S_{1} \{R\} \text{ and } \vdash_{p} \{\underbrace{wp(S_{2},Q)}_{R}\} S_{2} \{Q\}.$ Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr) 36 | 53 Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic From wp to proof trees: if-then-else $wp(\text{if } b \text{ then } S_1 \text{ else } S_2 \text{ fi}, Q) = (b \Rightarrow wp(S_1, Q)) \land (\neg b \Rightarrow wp(S_2, Q))$ Let $P = (b \Rightarrow wp(S_1, Q)) \land (\neg b \Rightarrow wp(S_2, Q))$. The proof tree is an instance of the inference rule for if-then-else: $\frac{\{b \land P\} S_1 \{Q\} \{\neg b \land P\} S_1 \{Q\}}{\{P\} \text{ if } b \text{ then } S_1 \text{ else } S_2 \text{ fi } \{Q\}}$ The premises have valid proof trees because: \blacktriangleright $b \wedge P$ is equivalent to $wp(S_1, Q)$ and by induction hypothesis, $\vdash_{p} \{wp(S_{1}, Q)\} S_{1} \{Q\}.$ ▶ $\neg b \land P$ is equivalent to $wp(S_2, Q)$ and by induction hypothesis, $\vdash_{p} \{wp(S_{2}, Q)\} S_{2} \{Q\}.$

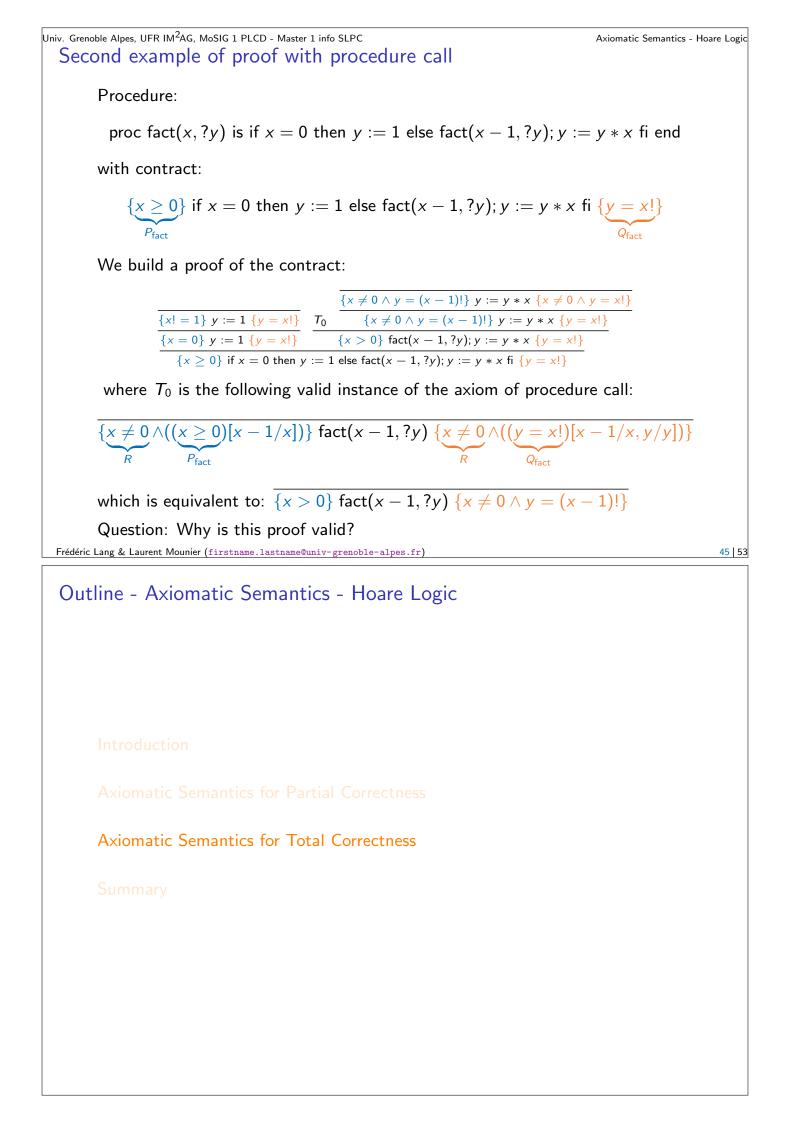
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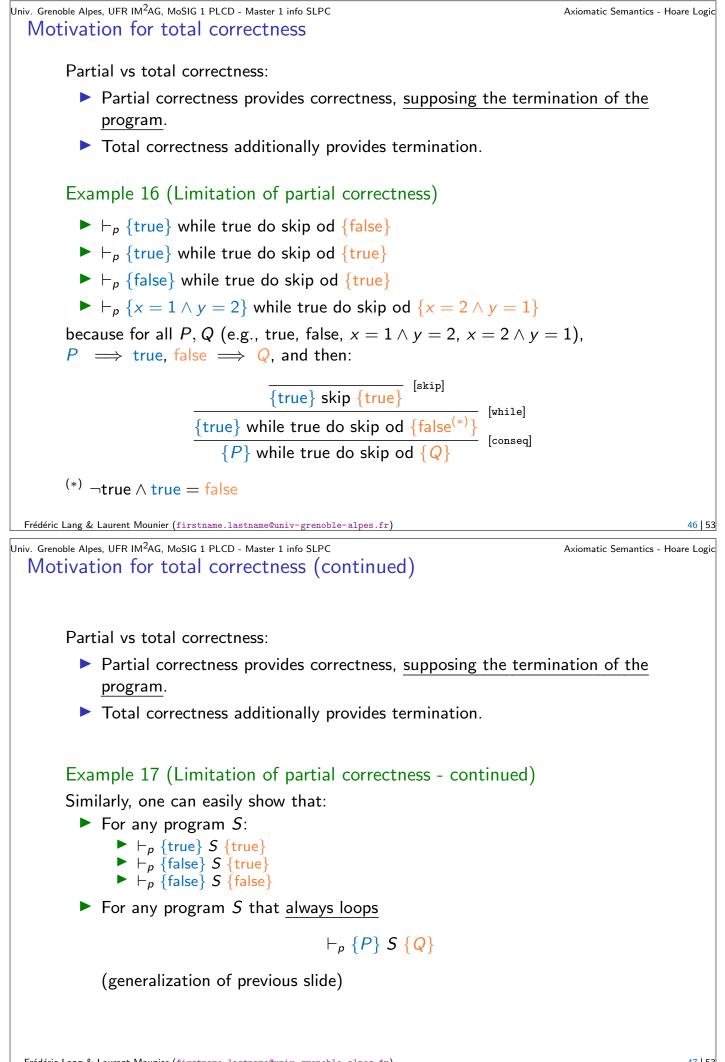


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	variants from your cou ram is correct, the inv	urses on algorithmics variant exists.		
Hints:				
Obser	ve how variables evolv	ve after a few iterations.		
		value of those variables, fo	r instance in	
	on of the number <i>n</i> o			
		<i>n</i> in terms of the program	variables and use it	
	predicate defined in p			
		sing the resulting predicate	as invariant	
5	ails, analyze what is w	o o 1		
	, ,	to be strengthen (e.g., adding	g a sign condition) if	
		halting condition does not im		
► Т	The invariant may have	to be weakened if it is not im		
•	recondition.	с., и.,		
-	•	f, the precondition, postcor	ndition, and/or	
progra	am may be incorrect!	Find a counterexample.		
	punier (firstname.lastname@univ-g			
oble Alpes, UFR IM	² AG, MoSIG 1 PLCD - Master 1 info		Axiomatic Semantics -	Ноа
mple: find	· · · ·		Axiomatic Semantics -	Ноа
mple: find Goal:	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant	SLPC		Ноа
bile Alpes, UFR IM ² mple: find Goal: $\{x = a \land a\}$	2^{2} AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \geq 0 \land y = 1$ while x	x > 0 do y := 2 * y; x := x		Ноа
bile Alpes, UFR IM mple: find Goal: $\{x = a \land a \in I\}$ iteration	$2^{AG, MoSIG 1 PLCD - Master 1 info}$ ding the invariant $y \ge 0 \land y = 1$ while x value of x	slpc x > 0 do y := 2 * y; x := x value of y		Hoa
bile Alpes, UFR IM ² mple: find Goal: $\{x = a \land a\}$	$2^{AG, MoSIG 1 PLCD - Master 1 info}$ ding the invariant $y \ge 0 \land y = 1$ while x value of x	slpc x > 0 do y := 2 * y; x := x value of y		Ноа
bile Alpes, UFR IM mple: find Goal: $\{x = a \land a \in \mathbb{R} \}$ iteration 0	$2^{AG, MoSIG 1 PLCD - Master 1 info}$ ding the invariant $y \ge 0 \land y = 1$ while x value of x	slpc x > 0 do y := 2 * y; x := x value of y		Ноа
bile Alpes, UFR IM mple: find Goal: $\{x = a \land a \}$ iteration 0 1	$2^{AG, MoSIG 1 PLCD - Master 1 info}$ ding the invariant $y \ge 0 \land y = 1$ while x value of x	slpc x > 0 do y := 2 * y; x := x value of y		Ноа
bile Alpes, UFR IM mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2	$2^{AG, MoSIG 1 PLCD - Master 1 info}$ ding the invariant $y \ge 0 \land y = 1$ while x value of x	x > 0 do y := 2 * y; x := x		Hoa
puble Alpes, UFR IM ² mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2 3 	$2^{AG, MoSIG 1 PLCD - Master 1 info}$ ding the invariant $y \ge 0 \land y = 1$ while x value of x	SLPC SLPC x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ 		Ноә
bile Alpes, UFR IM mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2 3 After	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - 1$	subset of $y := 2 * y; x := x$ value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$.	$-1 \text{ od } \{y = 2^a\}$	Ноэ
bille Alipes, UFR IM ² mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2 3 After From	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - x$ x = a - n, we get $n = a$	subset of $y := 2 * y; x := x$ value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ <i>n</i> and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$	$-1 \text{ od } \{y = 2^a\}$	Hoa
bile Alpes, UFR IM mple: find Goal: $\{x = a \land a$ iteration 0 1 2 3 After From Check	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - a$ x = a - n, we get $n = asing the precondition:$	subset of $y := 2 * y; x := x$ value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$.	$-1 \text{ od } \{y = 2^a\}$	Hoa
puble Alpes, UFR IM ² mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2 3 After From Check $2^{a-x} =$	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - 1$ x = a - n, we get $n = 1sing the precondition:x = 2^{a-a} = 2^0 = 1 = y.$	SLPC slpc x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y	$-1 \text{ od } \{y = 2^a\}$	Ног
bile Alpes, UFR IM mple: find Goal: $\{x = a \land a$ iteration 0 1 2 3 After From Check $2^{a-x} =$ Check	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - a$ x = a - n, we get $n = asing the precondition:= 2^{a-a} = 2^0 = 1 = y.Sing the postcondition$	SLPC slpc x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y $x \le 0$ (on halting) and y	$-1 \text{ od } \{y = 2^a\}$ $-x$ $x = 2^{a-x} \text{ as}$ $y = 2^{a-x} \text{ do not}$	Hoa
puble Alpes, UFR IM ² mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2 3 After From Check $2^{a-x} =$ Check imply	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - x$ x = a - n, we get $n = xsing the precondition:= 2^{a-a} = 2^0 = 1 = y.Sing the postcondition the postcondition y = a$	SLPC x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y $x \le 0$ (on halting) and y $= 2^a$. Must add constraint	$-1 \text{ od } \{y = 2^{a}\}$ $x = 2^{a-x} \text{ as}$ $y = 2^{a-x} \text{ do not}$ $x \ge 0.$	Ног
puble Alpes, UFR IM ² mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2 3 After From Check $2^{a-x} =$ Check imply	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - x$ x = a - n, we get $n = xsing the precondition:= 2^{a-a} = 2^0 = 1 = y.Sing the postcondition the postcondition y = a$	SLPC slpc x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y $x \le 0$ (on halting) and y	$-1 \text{ od } \{y = 2^{a}\}$ $x = 2^{a-x} \text{ as}$ $y = 2^{a-x} \text{ do not}$ $x \ge 0.$	Ног
bile Alpes, UFR IM mple: find Goal: $\{x = a \land a$ iteration 0 1 2 3 After From Check $2^{a-x} =$ Check imply We can no	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $n \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - a$ x = a - n, we get $n = asing the precondition:= 2^{a-a} = 2^0 = 1 = y.Sing the postconditionthe postcondition y = aw finish the proof using$	SLPC slpc x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y $x \le 0$ (on halting) and y $= 2^a$. Must add constraint ng the invariant $I \equiv y = 2^a$	$-1 \text{ od } \{y = 2^{a}\}$ $x = 2^{a-x} \text{ as}$ $y = 2^{a-x} \text{ do not}$ $x \ge 0.$	Hoa
pible Alpes, UFR IM mple: find Goal: $\{x = a \land a$ iteration 0 1 2 3 After From Check $2^{a-x} =$ Check imply We can no $\{y = 2^{a-x}, y\}$	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $p \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - 3$ n iterations, $x = a - 3$ n iterations, $x = a - 3$ x = a - n, we get $n = 3x = a - n$, we get $n = 3x = a - n$, we get $n = 3x = a - n$, we get $n = 3x = a - n$, we get $n = 3x = a^{-a} = 2^0 = 1 = y.Sing the postconditionthe postcondition y = 3w finish the proof usingx > 0 y := 2 * y \{y = 3$	SLPC slpc x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y x < 0 (on halting) and y $= 2^a$. Must add constraint ing the invariant $I \equiv y = 2^a$ $= 2^{a-x+1} \land x > 0$	$-1 \text{ od } \{y = 2^{a}\}$ $x = 2^{a-x} \text{ as}$ $y = 2^{a-x} \text{ do not}$ $x \ge 0.$ $x \ge 0.$	
pible Alpes, UFR IM mple: find Goal: $\{x = a \land a$ iteration 0 1 2 3 After From Check $2^{a-x} =$ Check imply We can no $\{y = 2^{a-x}, y\}$	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $p \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - 3$ n iterations, $x = a - 3$ n iterations, $x = a - 3$ x = a - n, we get $n = 3x = a - n$, we get $n = 3sing the precondition:= 2^{a-a} = 2^0 = 1 = y.So the postcondition y = 3w finish the proof usingx > 0 y := 2 * y \{y = 3x > 0$	SLPC SLPC x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y $x \le 0$ (on halting) and y $= 2^a$. Must add constraint ing the invariant $I \equiv y = 2^a$ $= 2^{a-x+1} \land x \ge 0$ $= 2^{a-x+1} \land x \ge 0$ $= 2^{a-x+1} \land x \ge 0$	$-1 \text{ od } \{y = 2^{a}\}$ $x = 2^{a-x} \text{ as}$ $y = 2^{a-x} \text{ do not}$ $x \ge 0.$ $y = x + 1 \land x \ge 0$	
poble Alpes, UFR IM mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2 3 After From Check $2^{a-x} =$ Check imply We can no $\{y = 2^{a-x}, y \}$	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - 3$ n iterations, $x = a - 3$ n iterations, $x = a - 3$ x = a - n, we get $n = 3Sing the precondition:= 2^{a-a} = 2^0 = 1 = y.Sing the postconditionthe postcondition y = 3w finish the proof using(x > 0) y := 2 * y \{y = 3(x > 0) y := 2 * y \{y = 3\}$	SLPC SLPC x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y x = a and $y = 1$ implies y $x = x \le 0$ (on halting) and y $= 2^a$. Must add constraint ng the invariant $I \equiv y = 2^a$ $= 2^{a-x+1} \land x \ge 0$ $= 2^{a-x+1} \land x \ge 0$ $= 2^{a-x+1} \land x \ge 0$ $= 2^{a-x+1} \land x \ge 0$	$-1 \text{ od } \{y = 2^{a}\}$ $x = 2^{a-x} \text{ as}$ $y = 2^{a-x} \text{ do not}$ $x \ge 0.$ $y = x \land x \ge 0:$ $y = x \land x \ge 0:$	
poble Alpes, UFR IM mple: find Goal: $\{x = a \land a \}$ iteration 0 1 2 3 After From Check $2^{a-x} =$ Check imply We can no $\{y = 2^{a-x}, y \}$	² AG, MoSIG 1 PLCD - Master 1 info ding the invariant $y \ge 0 \land y = 1$ while x value of x $x_0 = a$ $x_1 = x_0 - 1 = a - 1$ $x_2 = x_1 - 1 = a - 2$ $x_3 = x_2 - 1 = a - 3$ n iterations, $x = a - 3$ n iterations, $x = a - 3$ n iterations, $x = a - 3$ x = a - n, we get $n = 3Sing the precondition:= 2^{a-a} = 2^0 = 1 = y.Sing the postconditionthe postcondition y = 3w finish the proof using(x > 0) y := 2 * y \{y = 3(x > 0) y := 2 * y \{y = 3\}$	SLPC SLPC x > 0 do y := 2 * y; x := x value of y $y_0 = 1$ $y_1 = 2 * y_0 = 2 * 1$ $y_2 = 2 * y_1 = 2 * 2 * 1$ $y_3 = 2 * y_2 = 2 * 2 * 2 * 1$ n and $y = 2^n$. $= a - x$. Therefore, $y = 2^a$ x = a and $y = 1$ implies y $x \le 0$ (on halting) and y $= 2^a$. Must add constraint ing the invariant $I \equiv y = 2^a$ $= 2^{a-x+1} \land x \ge 0$ $= 2^{a-x+1} \land x \ge 0$ $= 2^{a-x+1} \land x \ge 0$	$-1 \text{ od } \{y = 2^{a}\}$ $x = 2^{a-x} \text{ as}$ $y = 2^{a-x} \text{ do not}$ $x \ge 0.$ $y = x \land x \ge 0:$ $y = x \land x \ge 0:$	









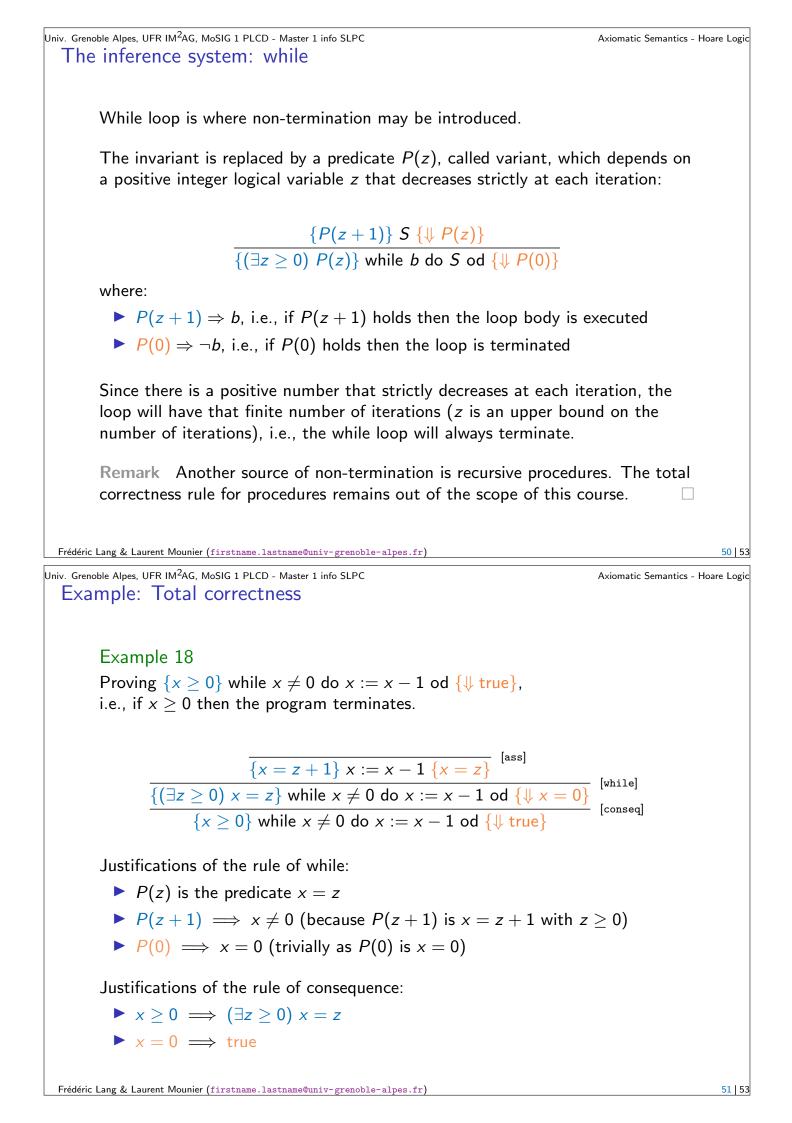
Total correctness assertions Triples of the form: $\{P\} S \{ \Downarrow Q \}$ if the precondition P is fulfilled S is guaranteed to terminate (\Downarrow) then the state after executing S satisfies the postcondition Qand Inference of Hoare triples $\vdash_t \{P\} S \{ \Downarrow Q \}$ Validity of Hoare triples $\models_t \{P\} S \{\Downarrow Q\}$ iff $\forall \sigma \in \mathbf{State} : P(\sigma)$ implies $(\exists \sigma' \in \mathbf{State}) \begin{cases} Q(\sigma') = \mathbf{tt} \\ (S, \sigma) \to \sigma' \end{cases}$ Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr) 48 | 53 Univ. Grenoble Alpes, UFR IM²AG, MoSIG 1 PLCD - Master 1 info SLPC Axiomatic Semantics - Hoare Logic The inference system: terminating statements Those rules are the same as for partial correctness as they may not themselves involve non-termination. Skip: $\{P\}$ skip $\{\Downarrow, P\}$ Assignment: $\{P[a/x]\} x := a \{ \Downarrow P \}$ Sequential composition: $\frac{\{P\} S_1 \{\Downarrow Q\} \{Q\} S_2 \{\Downarrow R\}}{\{P\} S_1 S_2 \{\Downarrow R\}}$ Conditional statement: $\frac{\{b \land P\} S_1 \{\Downarrow Q\} \{\neg b \land P\} S_2 \{\Downarrow Q\}}{\{P\} \text{ if } b \text{ then } S_1 \text{ else } S_2 \text{ fi } \{\Downarrow Q\}}$ Consequence: If $P \Rightarrow P'$ and $Q' \Rightarrow Q$, then:

$$\frac{\{P'\} S \{\Downarrow Q'\}}{\{P\} S \{\Downarrow Q\}}$$

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Axiomatic Semantics - Hoare Logic



Univ. Grenoble Alpes, UFR IM ² AG, MoSIG 1 PLCD - Master 1 info SLPC Properties of Hoare logic for total correctness	Axiomatic Semantics - Hoare Logic
Correctness (We can infer <u>only</u> valid triples)	
For every total correctness formula $\{P\} S \{ \Downarrow Q \}$, we have:	
If $\vdash_t \{P\} S \{\Downarrow Q\}$ then $\models_t \{P\} S \{\Downarrow Q\}$	
Completeness (We can infer <u>all</u> valid triples)	
For every total correctness formula $\{P\} S \{\Downarrow Q\}$, we have:	
If $\vDash_t \{P\} S \{\Downarrow Q\}$ then $\vdash_t \{P\} S \{\Downarrow Q\}$	
Relation between partial and total correctness For every assertion P, Q , and statement S, we have:	
If $\vdash_t \{P\} S \{\Downarrow Q\}$ then $\vdash_p \{P\} S \{Q\}$	
Remark The converse property does not hold.	
Frédéric Lang & Laurent Mounier (firstname.lastname@univ-grenoble-alpes.fr)	52 53
Outline - Axiomatic Semantics - Hoare Logic	
Summary	

Axiomatic Semantics

- Partial vs total correctness of programs.
- ► Focus on the essential properties.
- ► Hoare triples are assertions on programs.
- ► Hoare calculus inference system.
- Sound and complete.

Applications: design by contract and its implementations (e.g., JML for Java, D and in/out blocks), computer-aided proof of programs, B method.