Programming Language Semantics and Compiler Design

(Sémantique des Langages de Programmation et Compilation)

Maths Reminders

Yliès Falcone
ylies.falcone@univ-grenoble-alpes.fr — www.ylies.fr
Univ. Grenoble Alpes, and LIG-Inria team CORSE

Master of Sciences in Informatics at Grenoble (MoSIG)
Master 1 info

Univ. Grenoble Alpes - UFR IM^{2}AG
www.univ-grenoble-alpes.fr — im2ag.univ-grenoble-alpes.fr

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Some proof techniques

Proofs by contradiction, reducto-ad-absurdum, contraposition,. . .

. . . they rely on the principles of propositional and predicate logics.

Proof by structural induction

- Proof for the basic elements, atoms, of the set.
- Proof for composite elements (created by applying) rules:
  - assume it holds for the immediate components (induction hypothesis)
  - prove the property holds for the composite element

Induction on the shape of a derivation tree

- Proof for 'one-rule' derivation trees, i.e., axioms.
- Proof for composite trees:
  - For each rule R, consider a composite tree where R is the last rule applied
  - Proof for the composite tree
    - Assume it holds for subtrees, or premises of the rule (induction hypothesis)
    - Proof for the composite tree

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Proof by induction

Proving a predicate $P(n)$ that depends on some parameter $n \in \mathbb{N}$.

Example 1 (Predicate)

- $P(n) =$ The sum of the n natural numbers is $\frac{n \times (n+1)}{2}$."
- $P(n) =$ "If $q \geq 2$, we have $n \leq q^n$.
- $P(n) =$ "Every polynomial of degree n has at most n roots."

We want to prove $\forall n \in \mathbb{N} : P(n)$.

Principle

- Prove the **base case**: prove that $P(0)$ holds (or $P(k)$ if the minimal value of the parameter is k).
- Prove the **induction step**: prove that if for some n, $P(n)$ holds, then $P(n+1)$ holds.

The principle of induction ensures that $P(n)$ holds, for any $n \geq k$.

Proof by induction (example)

Example 2 (Proof by induction)

Let's prove that

$$\forall n \in \mathbb{N} : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

- Base case: $\sum_{i=0}^{n} i = 0$.
- Induction step.
  - Suppose that the property holds for some $n \in \mathbb{N}$.
  - We have:

$$\begin{aligned} \sum_{i=0}^{n+1} i &= \sum_{i=0}^{n} i + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \quad \text{(induction hypothesis)} \\ &= \frac{(n+2) \times (n+1)}{2} \end{aligned}$$

Proof by complete induction

Proving a predicate $P(n)$ that depends on some parameter $n \in \mathbb{N}$.

That is, we want to prove $\forall n \in \mathbb{N} : P(n)$.

Principle of complete induction

- Prove the **base case**: prove that $P(0)$ holds (or $P(k)$ if the minimal value of the parameter is k).
- Prove the **complete induction step**: prove that if for some n, $\forall m \leq n : P(m)$ holds, then $P(n+1)$ holds.

The principle of induction ensures that $P(n)$ holds for any $n \geq k$.

Proof by complete induction: example

The n-th fibonacci number f_n is defined as follows:

- $f_0 = 0$
- $f_1 = 1$
- $\forall k \geq 2 : f_k = f_{k-1} + f_{k-2}$

Let us prove that the n-th Fibonacci number is even iff n is a multiple of 3.

Example 3 (The n-th Fibonacci number is even iff n is a multiple of 3.)

- Base case. We can see that it holds for $n = 0$.
- complete induction step.
  - Let us suppose that the property holds for any integer lesser than or equal to some $n \in \mathbb{N}$.
  - Consider f_{n+1} and distinguish three cases according to the rest of the division of $n+1$ by 3.
    - Case $n+1 \mod 3 = 0$. $f_{3k} = f_{3k-1} + f_{3k-2} =$ odd + odd = even
    - Case $n+1 \mod 3 = 1$. $f_{3k+1} = f_{3k} + f_{3k-1} =$ even + odd = odd
    - Case $n+1 \mod 3 = 2$. $f_{3k+2} = f_{3k+1} + f_{3k} =$ odd + even = odd

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Proof by induction

Proof by structural induction

A notation: derivation tree

Inductive/Compositional definitions

Let us consider:

- E a set ,
- $f : E \times E \times \ldots \times E \rightarrow E$ a partial function,
- $A \subseteq E$ a subset of E.

Definition 4 (closure)

A is closed by f iff $f(A \times \ldots \times A) \subseteq A$.

Definition 5 (Construction rule)

A construction rule for a set states either:

- that a *basis element* belongs to the set, or
- how to produce a new element from existing elements (*production rule* given by a partial function).

Definition 6 (Inductive definition)

An inductive definition on E is a family of rules defining the smallest subset of E that is *closed* by these rules.

Inductive definitions: examples

Example 7 (Natural numbers)

How can define them?

- basis element 0
- 1 rule: $x \mapsto succ(x)$

2 is the natural number defined as $succ(succ(0))$

Example 8 (Even numbers)

- basis element 0
- 1 rule $x \mapsto x + 2$

Example 9 (Palindromes on $\{a, b\}$)

- basis elements ϵ, a, b
- 2 rules: $w \mapsto a \cdot w \cdot a$, $w \mapsto b \cdot w \cdot b$

Binary trees

Definition and examples

Definition 10 (Binary Tree – Informal definition)

A tree is a binary tree if each node has *at most two children* (possibly empty).

Definition 11 (Binary Tree – Mathematical definition)

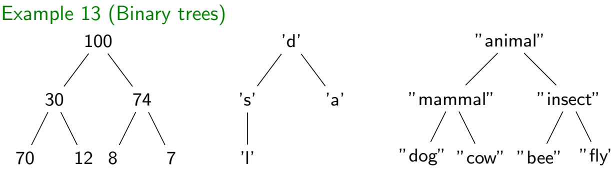
The smallest set $Bt(Elt)$ s.t.:

$$Bt(Elt) = \{EmptyT\} \cup \{Node(tL, e, tR) \mid e \in Elt \wedge tL, tR \in Bt(Elt)\}$$

Example 12 (Binary trees of natural numbers)

$$Bt(\mathbb{N}) = \{EmptyT\} \cup \{Node(tL, e, tR) \mid e \in \mathbb{N} \wedge tL, tR \in Bt(\mathbb{N})\}$$

Example 13 (Binary trees)



Proof by Structural Induction

Proving that the proof holds for any element "however it is built".

Principle

- Proof for the basic elements, atoms, of the set.
- Proof for composite elements (created by applying) rules:
  - assume it holds for the immediate components (induction hypothesis)
  - prove the property holds for the composite element

Proof by Structural Induction: example

Example 14 (Proofs by induction)

All proofs by induction are proofs by structural induction where the inductive set is $\mathbb{N}$.

Example 15 (Properties of size and depth of a binary tree)

Let us consider $t \in Bt(Elt)$, a binary tree:

- $\mathrm{depth}(t)$ be the depth of tree t: length of longest path from root to leaf.
- $\mathrm{size}(t)$ be the size of tree t: number of nodes + leaves.

For any type Elt and any $t \in Bt(Elt)$:

- $\mathrm{depth}(t) \leq \mathrm{size}(t)$,
- $\mathrm{size}(t) \leq 2^{\mathrm{depth}(t)-1}$.

Inductive definitions: examples

Example 16 (Natural numbers)

How can define them?

- basis element 0
- 1 rule: $x \mapsto succ(x)$

2 is the natural number defined as $succ(succ(0))$

Example 17 (Even numbers)

- basis element: 0;
- 1 rule: $x \mapsto x + 2$.

Example 18 (Palindromes on $\{a, b\}$)

- basis elements: ϵ, a, b;
- 2 rules: $w \mapsto a \cdot w \cdot a$, $w \mapsto b \cdot w \cdot b$.

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A notation: derivation tree

Notation for $t = f(x_1, \dots, x_n)$

$$\frac{x_1 \quad \dots \quad x_n}{t}\ f$$

"t is built/obtained from $x_1, \dots, x_n$" by applying operator f.

Example 19 (Derivation trees)

- $2 = \mathrm{succ}(\mathrm{succ}(\mathrm{succ}(0)))$ is a natural number

$$\cfrac{\cfrac{\overline{0}}{1}}{2}\ \mathrm{succ}$$

- *aba* is a palindrome:

$$\frac{\overline{b}}{aba}$$

- *ababa* is a palindrome:

$$\cfrac{\cfrac{\overline{a}}{bab}}{ababa}$$

Abstract syntax trees and derivation trees

Consider an abstract syntax tree, produced by syntactic analysis.

Derivation tree

For each node, computing information from the information of its sons.

Example 20 (Derivation trees in type analysis)

We obtain for each node a type (or error) based on types of its sons.

Definition of derivation trees by a formal system

Generally, we have information, stored in some environment Γ. The formal system states how to *deduce* knowledge from existing knowledge, where knowledge is of the form $\Gamma \vdash \mathcal{P}$, which means $\mathcal{P}$ holds on Γ.

- a set of axiom schemes
- a set of inference rules : a rule of the form

$$\frac{\Gamma_1 \vdash \mathcal{P}_1 \quad \cdots \quad \Gamma_n \vdash \mathcal{P}_n}{\Gamma \vdash \mathcal{C}}$$

"if the hypothesis (premisses) $\mathcal{P}_i$ hold, then the conclusion $\mathcal{C}$ holds.