Tomography and 3D reconstruction from 2D projections

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Contents

List of Figures

Chapter 1

Introduction

CT Scanners and nuclear imaging (SPECT and PET) have greatly improved medical diagnoses and surgical planning. Mathematics is necessary for these medical imaging systems to work. We present mathematical problems arising from these medical imaging systems. We show how to reconstruct images from projections of the attenuation function in radiology or respectively of the activity in nuclear imaging. We present recent advances in 2D and 3D reconstruction problems.

This presentation covers 2D tomography including the reconstruction of Region Of Interest from non-complete data (very short scan trajectories, truncated projections), 3D tomography from Orlov and Tuy Conditions to Kolsher filter and Katsevich reconstruction formula, and dynamic tomography.

In this introduction we present briefly the physical interactions between photons and matter. We derive the mathematical problem formulation of a function reconstruction from its projections. We introduce the Radon transform and the x-ray transform, their basic properties, in particular the Fourier slice theorem. In 2D tomography, we show the Filtered BackProjection inversion formula and its application to fan-beam geometries. We then concentrate on recent advances in ROI reconstruction from incomplete projections. In 3D reconstruction, we derive inversion conditions and formulas for the parallel geometry and the Cone Beam Geometry. We develop recent advances based on the Katsevich formula. We then introduce dynamics problems, i.e. reconstruction from dynamic object. We consider the problem of reconstructing a 2D dynamic object from its projections and show extensions to 3D dynamic reconstruction.

1.1 Some historical facts, aims of the book

Reconstruction mehods of a function from its integral values on lines were proposed for applications in astronomy [2] as early as the fifties. But the first inversion is due to J. Radon [17] in 1917 in a general mathematical framework. Even if X-rays were already discovered in 1895 by W. Röntgen[1], no tomographic application was proposed before begin of the sixties. Cormack [4], proposed methods and algorithms to reconstruct the interior densities of an object from exterior x-ray attenuation measurements. Hounsfield [12] constructed the first medical CT. Both Cormack and Hounsfield received the Nobel price for their pionneering work in 1979.

The aim of the book is fist to give an introduction to the mathematics of tomography. We also want to underline the results made since the early 2000. In 2002, at the conference "Fully 3D image reconstruction", the Katsevitch formula establishing a analytic inversion of the 3D conical transform on the helix was the starting points for a decade of new developments in 2D and 3D reconstruction. His formula, was essentially a weighted filtered back-projection, with the filtering occurring on lines on the detector. He generalized soon the formula to other source trajectories. His work influenced also greatly the way to consider 2D reconstruction problems. This new and modern way to solve the reconstruction problem yields ROI approaches.

The notations are mainly inspired on the excellent book [15].

1.2 Radiology and nuclear imaging, physical models

1.2.1 Radiology

1.2.2 Nuclear Imaging

1.2.3 Other medical imaging modalities

1.2.4 Application context

1.3 Notations, standard results

We consider the attenuation function μ of an object in $\mathbb{R}^n, d \in \mathbb{N}, d \geq 2$. In practice we restrict the dimension n to $n = 2$ for single slice tomography, $n = 3$ for 3D tomography or for 2D dynamic tomography and $n = 4$ for 3D dynamic tomography. We suppose that $\mu : \mathbb{R}^n \to \mathbb{R}$ is a bounded function of compact support contained in Ω, where $\Omega = \{\vec{x} \in \mathbb{R}^n, \|\vec{x}\| \leq 1\}$ and $\|\vec{x}\| = \sqrt{\sum_{j=1}^{d} x_i^2}$ is the euclidean norm of $\vec{x}$. Thus $\mu \in L^p(\mathbb{R}^n), \forall p \in \mathbb{N}^\star$.

[1] Über eine neue Art von Strahlen (On a New Kind of Rays), 28 december 1995 application for publication of W. C. Röntgen at the "Physikalisch-medizinische Gesellschaft Würzburg" (physical-medical association)

1.3.1 Radon transform

Let $\mu \in L^1(\mathbb{R}^n)$ then the Radon transform $\mathcal{R}\mu$ of μ is defined by

$$\mathcal{R}\mu\left(\vec{\theta}, s\right) \stackrel{\text{def}}{=} \int_{\vec{\theta}^{\perp}} \mu\left(s\vec{\theta} + \vec{y}\right) d\vec{y} = \int_{\vec{x}\cdot\vec{\theta}=s} \mu(\vec{x})\, d\vec{x} = \int_{\mathbb{R}^n} \mu(\vec{x})\, \delta\left(\vec{x}\cdot\vec{\theta} - s\right) d\vec{x} \tag{1.1}$$

where $\vec{\theta} \in \mathcal{S}^{d-1}$, the unit sphere in $\mathbb{R}^n$, $\vec{\theta}^{\perp}$ is the hyperplane orthogonal to $\vec{\theta}$. We define the Radon projection in the direction $\vec{\theta}$ by

$$\mathcal{R}_{\vec{\theta}}\mu(s) \stackrel{\text{def}}{=} \mathcal{R}\mu\left(\vec{\theta}, s\right) \tag{1.2}$$

From Fubini theorem we have immediately that $\left(\mathcal{R}_{\vec{\theta}}\mu\right) \in \mathbb{L}^p(\mathbb{R})$.

1.3.2 X-ray transforms

Let $\mu \in L^1(\mathbb{R}^n)$ then the X-ray transform $\mathcal{X}\mu$ of μ is defined by

$$\mathcal{X}\mu\left(\vec{\zeta}, \vec{y}\right) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \mu\left(\vec{y} + l\vec{\zeta}\right) dl \tag{1.3}$$

where $\vec{\zeta} \in \mathcal{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$, $\vec{y} \in \vec{\zeta}^{\perp}$ is in the hyperplane orthogonal to $\vec{\zeta}$.

1.3.3 Divergent Beam transform

Let $\vec{v} \in \mathbb{R}^n$ denotes a point (an X-ray source position in X-ray CT), and $\vec{\zeta} \in \mathbb{S}^{n-1}$ a unit vector (in the direction from the source to the detector in X-ray CT), the Divergent Beam transform is defined by

$$\mathcal{D}\mu\left(\vec{v}, \vec{\zeta}\right) = \int_0^{+\infty} \mu\left(\vec{v}_t + l\vec{\zeta}\right) dl \tag{1.4}$$

Generally (in particular in X-ray CT) the data are acquired from multiple source positions and the source describe a trajectory along a curve

$$\begin{array}{rcl} \vec{v} \;:\; T \subset \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ t & \longrightarrow & \vec{v}(t) \end{array}$$

We will suppose in the following that the source trajectory $\mathcal{C} = \{v(t), t \in T\}$, is outside of Ω (outside of the convex hull of the support of μ is a sufficient condition), i.e., $\Omega \cap \mathcal{C} = \emptyset$. In practice, the source trajectory is sampled. The number $p \in \mathbb{N}$ of x ray projections is bounded. Thus we deal with a finite number of vertices, $\vec{v}_i \in \mathbb{R}^n, i = 1, \ldots, p$ (and $\vec{v}_i = \vec{v}(t_i), t_i \in T$ is the sampling of the source trajectory).

In 2D, the fan beam data can be parametrized

$$d(t, \alpha) = \mathcal{D}\mu\left(\vec{v}(t), \vec{\zeta}(\alpha)\right) \tag{1.5}$$

with $\vec{\zeta}(\alpha) = (-\sin\alpha, \cos\alpha)$.

Chapter 2

2D Tomography

2.1 Elementary properties of the Radon and the X-ray transforms

2.1.1 Parallel geometry

In the following we denote for $n = 2$ by p the parallel projections of a function μ on the plane ($\mu \in \mathbb{L}^1\left(\mathbb{R}^2\right)$):

$$p(\phi, s) \stackrel{\text{def}}{=} \mathcal{X}\mu\left(\vec{\zeta}, s\vec{\theta}\right) = \mathcal{R}\mu\left(\vec{\theta}, s\right) = \int_{\mathbb{R}} \mu\left(l\vec{\zeta} + s\vec{\theta}\right) \mathrm{d}l, \qquad (2.1)$$

where $\vec{\theta} = (\cos\phi, \sin\phi)\,, \vec{\zeta} = (-\sin\phi, \cos\phi)\,, \phi \in [0, 2\pi)$.

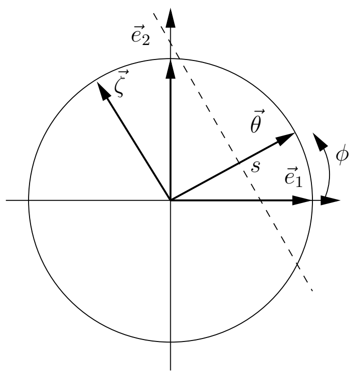

Figure 2.1: 2D tomography: parallel geometry parameters. The line of integration $s\vec{\theta} + \mathbb{R}\vec{\zeta}$ is the dashed line.

2.1.2 Elementary properties

We easily see the symetry of the Radon transform $p(\phi + \pi, s) = p(\phi, -s)$ (because $\vec{\theta}(\phi + \pi) = -\vec{\theta}(\phi)$ the line of equation $\vec{\theta}(\phi + \pi) \cdot \vec{x} = s$ is the same as $\vec{\theta}(\phi)\cdot\vec{x} = -s$). We define p at fixed ϕ by $p_\phi(s) \stackrel{\text{def}}{=} p(\phi, s)$. As $\mu \in \mathbb{L}^p(\mathbb{R}^2)$, by Fubini's theorem $p_\phi \in \mathbb{L}^p(\mathbb{R})$ thus $p \in \mathbb{L}^p_P(2\pi) \times \mathbb{L}^p(\mathbb{R})$[1]. Thus we can define the Fourier transform of p_ϕ

$$\hat{p}_\phi(\sigma) = \int_{\mathbb{R}} p_\phi(s) e^{-2i\pi\sigma s} \mathrm{d}s$$

and we have $p_\phi(s) = \int_{\mathbb{R}} \hat{p}_\phi(\sigma) e^{2i\pi\sigma s} d\sigma$ (at least in the $\mathbb{L}^2$ sense, more strongly for regular function, e.g. punctually when p_ϕ is continuous).

We can define the 2D Fourier transform of p by

$$\hat{p}_k(\sigma) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} p(\phi, s) e^{-2i\pi(\sigma s + k\phi)} \mathrm{d}s\mathrm{d}\phi$$

and

$$p(\phi, s) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{p}_k(\sigma) e^{+2i\pi(\sigma s + k\phi)} \mathrm{d}s$$

2.1.3 Projection slice theorem

Theorem 2.1.1. *Let $\mu \in \mathbb{L}^1(\mathbb{R}^2)$ then*

$$\widehat{\mathcal{R}_{\vec{\theta}}\mu}(\sigma) = \hat{\mu}(\sigma\vec{\theta})$$

Proof.

$$\begin{aligned}
\widehat{\mathcal{R}_{\vec{\theta}}\mu}(\sigma) &= \int_{\mathbb{R}} \mathcal{R}_{\vec{\theta}}\mu(s) e^{-2i\pi\sigma s}\mathrm{d}s \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mu\left(s\vec{\theta} + l\vec{\zeta}\right)\mathrm{d}l \right) e^{-2i\pi\sigma s}\mathrm{d}s \\
&= \int_{\mathbb{R}^2} \mu(\vec{x}) e^{-2i\pi\sigma\vec{\theta}\cdot\vec{x}} d\vec{x} \\
&= \hat{\mu}(\sigma\vec{\theta})
\end{aligned}$$

where we have made the change of variable (s, l) to (x_1, x_2) i.e. $\vec{x} = s\vec{\theta} + l\vec{\zeta}$. This is a rotation thus $\mathrm{d}l\mathrm{d}s = dx_1 dx_2$ □

Note that with our 2D notation this theorem reads $\hat{p}_\phi(\sigma) = \hat{\mu}(\sigma\vec{\theta})$.

2.1.4 Filtered Back Projection inversion formula

Theorem 2.1.2. *Let $\mu \in \mathbb{L}^1(\mathbb{R}^2)$ sufficiently smooth then*

$$\mu(\vec{x}) = \int_0^{\pi}\int_{\mathbb{R}} \widehat{\mathcal{R}_{\vec{\theta}}\mu}(\sigma)|\sigma| e^{2i\pi\sigma\vec{x}\cdot\vec{\theta}} d\sigma\mathrm{d}\phi$$

[1] $L^p_P(a)$ is the set of periodic functions f of period $a > 0$ such that the integral of $|f|^p$ over a period exists with $p > 0$

Proof.

$$\begin{aligned}
\mu(\vec{x}) &= \int_{\mathbb{R}^2} \hat{\mu}(\vec{\xi}) e^{2i\pi\vec{\xi}\cdot\vec{x}} d\vec{\xi} \\
&= \int_0^\pi \int_{\mathbb{R}} \hat{\mu}(\sigma\vec{\theta}) e^{2i\pi\sigma\vec{\theta}\cdot\vec{x}} |\sigma| d\sigma \mathrm{d}\phi \\
&= \int_0^\pi \int_{\mathbb{R}} \widehat{\mathcal{R}_{\vec{\theta}}\mu}(\sigma) |\sigma| e^{2i\pi\sigma\vec{\theta}\cdot\vec{x}} d\sigma \mathrm{d}\phi
\end{aligned}$$

where we have made the polar change of variables $\vec{\xi} = \sigma\vec{\theta}(\phi)$, $(\phi, \sigma) \in [0\pi[\times\mathbb{R}$, thus $d\xi_1 d\xi_2 = |\sigma| d\sigma \mathrm{d}\phi$, and we have used theorem 2.1.1. □

Note that with our 2D notation this theorem reads

$$\mu(\vec{x}) = \int_0^\pi \int_{\mathbb{R}} \hat{p}_\phi(\sigma) |\sigma| e^{2i\pi\sigma\vec{x}\cdot\vec{\theta}} d\sigma \mathrm{d}\phi.$$

From therorem 2.1.2 we can derive by a simple discretization of both integrals an algorithm known as the Filtered Back Projection (FBP) algorithm, see [15]. This formula is generally decomposed into two steps

1. The **filtering step** by the ramp filter

   $$\forall\phi, p_F(\phi, s) = \int_{\mathbb{R}} \hat{p}_\phi(\sigma) |\sigma| e^{2i\pi\sigma s} d\sigma$$

   Generally the filtering in the Fourier space $|\sigma|$ called the ramp filtering (denoted by r here) is replaced by more regular (and less sensitive to noise) filters such as $\hat{r}_c(\sigma) = \chi_{[-c,c]}(\sigma)|\sigma|$, where $c > 0$ is a cut-off frequency, i.e,.

   $$r_c(s) = \int_{-c}^{c} \chi_{[-c,c]}(\sigma) |\sigma| e^{2i\pi\sigma s} d\sigma$$

   Thus p_F is the convolution of p_ϕ by r, i.e. $p_F(\phi, s) = r_c \star p_\phi(s)$, more precisely

   $$p_F(\phi, s) = \int_{\mathbb{R}} r_c(s - u) p_\phi(u) \mathrm{d}u.$$

   Note thas r_c is a non-local filter with long dependancy. See also [15] for other filters such as the Shepp and Logan filter.

2. The filtering step is followed by the **back projection** step

   $$\mu(\vec{x}) = \int_0^\pi p_F(\phi, \vec{x}\cdot\theta) \mathrm{d}\phi.$$

We remark that $|\sigma| = \frac{1}{2\pi}(2i\pi\sigma)\left(-i\mathrm{sgn}(\sigma)\right)$. Thus the filter is the Hilbert filtering (multiplication of by $\hat{p}_\phi(\sigma)$ by $-i\mathrm{sgn}(\sigma)$) composed by the derivation (multiplication by $2i\pi\sigma$). The filtering step can be thus written $p_H(\phi, s) = Hp_\phi(s)$ see (2.3) followed by a derivation $p_F(\phi, s) = \frac{\partial p_H}{\partial s}(\phi, s)$. Here also a regularized Hilbert filtering is used in practice, see section A.2.1.

2.1.5 Fanbeam geometry

Source trajectories

We first define the source trajectory along a curve outside the convex hull of the support of μ

$$\begin{array}{rcll} \vec{v} & : & T & \longrightarrow & \mathbb{R}^2 \\ & & t & \longrightarrow & \vec{v}(t) \end{array}$$

The fan-beam data are then defined by

$$d\left(\vec{v}_t, \alpha\right) \stackrel{\text{def}}{=} \int_0^{+\infty} \mu\left(\vec{v}_t + l\vec{\zeta}(\alpha)\right) \mathrm{d}l \tag{2.2}$$

We remark that

$$p(\phi, s) = d(\vec{v}_t, \phi) + d(\vec{v}_t, \phi + \pi) \text{ where } s = \vec{v}_t \cdot \vec{\theta}(\phi).$$

In the following, we make the small abuse of notation $d\left(t, \alpha\right) \stackrel{\text{def}}{=} d\left(\vec{v}_t, \alpha\right)$.

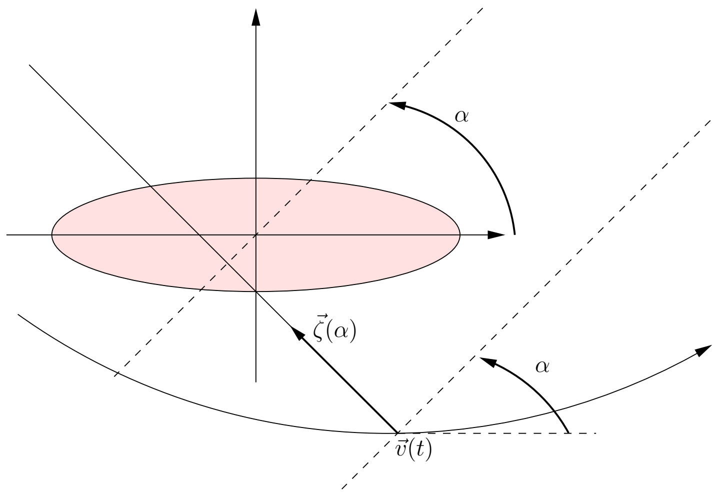

Figure 2.2: The Fan Beam variables (t, α)

Inversion formulas and Parker weights

We consider the circular trajectory, $\vec{v}_t = (-R_v \cos t, -R_v \sin t)$,

Theorem 2.1.3. *Let $\mu \in \mathbb{L}^1(\mathbb{R}^2)$ sufficiently smooth then*

$$\mu(\vec{x}) = \frac{1}{2} \int_0^{2\pi} \frac{1}{\|\vec{x} - \vec{v}_t\|^2} d_{WF}\left(\vec{v}_t, \arg(\vec{x} - \vec{v}_t)\right) \mathrm{d}t$$

where

$$d_{WF}(\vec{v}_t,\phi)=\int_{t-\pi/2}^{t+\pi/2} R_v\cos(\psi-t)d(\vec{v}_t,\psi)r\left(\sin(\phi-\psi)\right)\mathrm{d}\psi$$

where r is the ramp filter ($\hat{r}(\sigma)=|\sigma|$).

Proof. We start from the FBP formula on $[0,2\pi]$, writen under the convolution form

$$\mu(\vec{x})=\frac{1}{2}\int_0^{2\pi}\int_{\mathbb{R}} r(\vec{x}\cdot\vec{\theta}-s)p(\phi,s)\mathrm{d}s\mathrm{d}\phi,$$

where r is the ramp filter (in practice we use a regularized version). We see in Fig. 2.3 that $d(t,\alpha)=p\left(\alpha,\vec{v}(t)\cdot\vec{\theta}(\alpha)\right)$. Thus we want to change in the previous formula the variables s,ϕ into the variables t,α with $\phi=\alpha$ and $s=\vec{v}(t)\cdot\vec{\theta}(\alpha)$. For a given a given (ϕ,s) we remark that there exists one (t,α) such that $\alpha=\phi$ and $\vec{v}(t)\cdot\vec{\theta}(\phi)=s$ (with $\vec{\zeta}(\phi)\cdot\vec{v}(t)\le 0$ else the half line $\vec{v}(t)+\mathbb{R}^+\vec{\zeta}(\alpha)$ do not cross Ω) i.e. $-R(-\sin t\cos\phi+\cos t\sin\phi)=s$ i.e., $\sin(t-\phi)=s/R$ i.e. $t=\phi+\arcsin\left(\frac{s}{R}\right)$. The change of variable

$$\left\{\begin{array}{l}\phi=\alpha\\ s=R\sin(t-\alpha)\end{array}\right.\quad\text{inversely}\quad\left\{\begin{array}{l}t=\phi+\arcsin\left(\frac{s}{R}\right)\\ \alpha=\phi\end{array}\right.$$

is one to one for $(\phi,s)\in[0,2\pi)\times[-1,1]$ and $(t,\alpha)\in[0,2\pi)\times[\alpha_m(t),\alpha_M(t)]$ where $\alpha_m(t)=t-\arcsin\left(\frac{1}{R}\right)$, $\alpha_M(t)=t+\arcsin\left(\frac{1}{R}\right)$ ($\alpha_M-\alpha_m=2\arcsin\left(\frac{1}{R}\right)$ is the fan-angle).

The Jacobian matrix of the variable change is

$$J(t,\alpha)=\left[\begin{array}{cc}\frac{\partial\phi}{\partial t}&\frac{\partial\phi}{\partial\alpha}\\ \frac{\partial s}{\partial t}&\frac{\partial s}{\partial\alpha}\end{array}\right]=\left[\begin{array}{cc}0&1\\ R\cos(t-\alpha)&-R\cos(t-\alpha)\end{array}\right]$$

thus $|\det J|=R\cos(t-\alpha)$ (always positive because $\alpha-t\in[\alpha_m,\alpha_M]\subset[-\frac{\pi}{2},\frac{\pi}{2}]$. Thus we have

$$\mu(\vec{x})=\frac{1}{2}\int_0^{2\pi}\int_{\alpha_m(t)}^{\alpha_M(t)} R\cos(t-\alpha)r\left(\vec{x}\cdot\vec{\theta}(\alpha)-\vec{v}_t\cdot\theta(\alpha)\right)p(\alpha,\vec{v}_t\cdot\theta(\alpha))\mathrm{d}\alpha\mathrm{d}t,$$

We remark that

$$r\left(\vec{x}\cdot\vec{\theta}-\vec{v}_t\cdot\theta\right)=\frac{1}{\|\vec{x}-\vec{v}_t\|^2}r\left(\frac{\vec{x}-\vec{v}_t}{\|\vec{x}-\vec{v}_t\|}\cdot\vec{\theta}\right)$$

Let us now define ψ such that

$$\zeta(\psi)\stackrel{\text{def}}{=}\frac{\vec{x}-\vec{v}_t}{\|\vec{x}-\vec{v}_t\|}$$

i.e.,

$$\psi+\frac{\pi}{2}=\arg\left(\frac{\vec{x}-\vec{v}_t}{\|\vec{x}-\vec{v}_t\|}\right)$$

then

$$\frac{\vec{x}-\vec{v}_t}{\|\vec{x}-\vec{v}_t\|}\cdot\vec{\theta}(\alpha)=\sin(\alpha-\psi)$$

and as r is even we have

$$r\left(\frac{\vec{x}-\vec{v}_t}{\|\vec{x}-\vec{v}_t\|}\cdot\vec{\theta}\right)=r\left(\sin(\psi-\alpha)\right)$$

Thus

$$\mu(\vec{x})=\frac{1}{2}\int_0^{2\pi}\frac{1}{\|\vec{x}-\vec{v}(t)\|^2}d_F(\vec{v}_t,\psi)_{|\psi=\arg(\vec{x}-\vec{v}_t)-\frac{\pi}{2}}\mathrm{d}t$$

with

$$d_F(\vec{v}_t,\phi)=\int_{\alpha_m(t)}^{\alpha_M(t)}R\cos(t-\alpha)r(\sin(\phi))d(t,\alpha)\mathrm{d}\alpha,$$

□

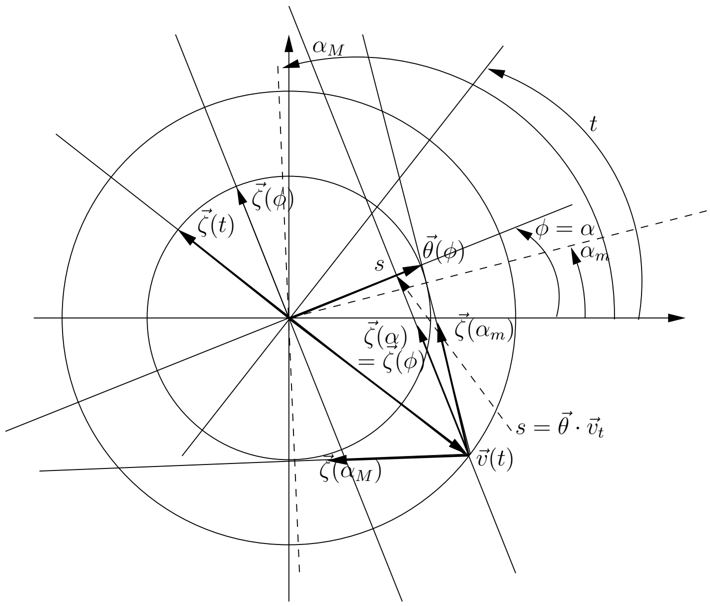

Figure 2.3: The parallel variables (ϕ, s) are changed to the fan beam variables (t,α) such that $s=\vec{\theta}\cdot\vec{v}_t$ and $\phi=\alpha$

2.2 Incomplete data

2.2.1 Definitions and classical results

Interior and exterior problems

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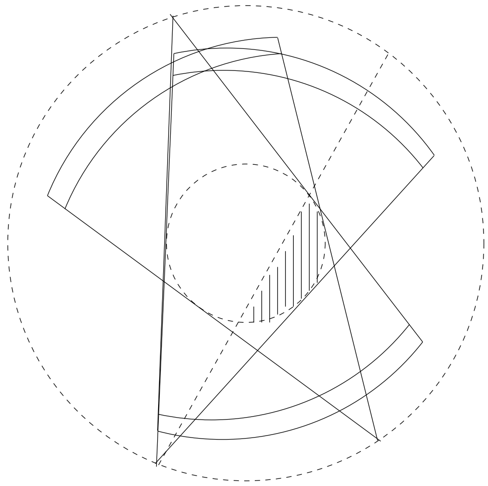

Figure 2.4: Short scan with (Parker) weight is possible....

limited angular data

truncated projections

2.3 ROI approaches

A very good review of 2D ROI reconstruction approaches has been presented in [3] and this section is mainly based on this review.

2.3.1 Hilbert Filtered Back Projection

The Hilbert Filtered Back Projection method is an indirect consequence of the Katsevich inversion formula for 3D helical cone beam tomography. Indeed, the Katsevich formula was in 2002 a deep innovation. It was tempted to study what was the consequence for 2D fan-beam problem examining what arise when the helical pitch was converging to zero. It was the starting point of [16] and the beginning of a rich series of papers on both 3D reconstruction and 2D ROI methods.

Introduction and notations

We define p_H the Hilbert transform of the parallel projection p

$$p_H(\phi, s) = \int_{-\infty}^{+\infty} p(\phi, u)h(s-u)\mathrm{d}u \text{ where } h(u) = \frac{1}{\pi u} \tag{2.3}$$

and $\hat{h}(\sigma) = -\imath \mathrm{sgn}(\sigma)$ (distribution). The ramp filtering p_R of p is

$$p_R(\phi, s) = \frac{1}{2\pi}\frac{\partial}{\partial s} p_H(\phi, s)$$

We define d_H the Hilbert transform of the fan beam projection d

$$d_H(\vec{v}_t,\phi) = \int_0^{2\pi} d_H(\vec{v}_t,\psi) h\left(\sin(\phi-\psi)\right) \mathrm{d}\psi \text{ where } h(u) = \frac{1}{\pi u} \tag{2.4}$$

The parallel Fan Beam Hilbert Projection Equality and inversion formula

Theorem 2.3.1 (Parallel Fan Beam Hilbert Projection Equality)**.**

$$p_H(\phi,s) = d_H(\vec{v}_t,\phi), \ \ \textit{where } s = \vec{v}_t \cdot \vec{\theta}(\phi) \tag{2.5}$$

Let us consider a point $\vec{x}$ in the plane belonging to the line (ϕ,s) i.e. $\vec{\theta}\cdot\vec{x} = s$. If there exists a source vertex $\vec{v}_t$ such that $s = \vec{v}_t \cdot \vec{\theta}(\phi)$ i.e.$\vec{v}_t \in L(\phi,s)$, i.e $\vec{v}_t + \mathbb{R}\vec{\zeta} = \vec{x} + \mathbb{R}\vec{\zeta} = s\vec{\theta} + \mathbb{R}\vec{\zeta}$, see Fig 2.5, then $p_H(\phi,s) = p_H(\phi,\vec{x}\cdot\vec{\theta})$ needed for the FBP reconstruction at $\vec{x}$ can be computed from $g(\vec{v}_t,\psi), \psi \in (-\pi/2,\pi/2)$ as soon as $g(\vec{v}_t,\cdot)$ is not truncated, even if $p_H(\phi,\cdot)$ is truncated.

Proof. We suppose that the line (ϕ,s) contains the vertex $\vec{v}_t$, i.e., $\vec{v}_t\cdot\vec{\theta} = s$. Without loss of generality, we translate the origin of the referential so that $\vec{v}_t$ is the origin, i.e. $\vec{v}_t = 0$ and we rotate so that, $\phi = 0$. We have with $\phi = 0$ and $s = 0$ in Eq. (2.3)

$$\begin{aligned} p_H(0,0) &= \int_{\mathbb{R}} p(0,t)h(-t)\mathrm{d}t = \int_{\mathbb{R}^2} \mu\left(t(1,0)+l(0,1)\right)\mathrm{d}l h(-t)\mathrm{d}t \\ &= \int_{\mathbb{R}^2} \frac{\mu\left(t,l\right)}{-\pi t}\mathrm{d}l\mathrm{d}t \end{aligned}$$

We also have with $\vec{v}_t = (0,0)$ and $\phi = 0$ in Eq. (2.4)

$$\begin{aligned} d_H\left(\vec{v}_t,\phi\right) &= d_H\left((0,0),0\right) \\ &= \int_0^{2\pi} g\left((0,0),\psi\right) h\left(\sin(-\psi)\right)\mathrm{d}\psi \\ &= \int_0^{2\pi}\int_0^{+\infty} \mu\left((0,0)+l\vec{\zeta}(\psi)\right)\mathrm{d}l h\left(\sin(-\psi)\right)\mathrm{d}\psi \\ &= \int_0^{2\pi}\int_0^{+\infty} \frac{\mu\left(l\cos(\psi),l\sin(\psi)\right)l}{-\pi\sin(\psi)l}\mathrm{d}l \end{aligned}$$

Let us make the polar to cartesian change of variables $(r,t) = (l\cos\psi, l\sin\psi)$, $\mathrm{d}r\mathrm{d}t = |l|\mathrm{d}l\mathrm{d}\psi$, $(l,\psi) \in \mathbb{R}^+ \times [0,\pi)$, $(r,t) \in \mathbb{R}^2$.

$$d_H\left(\vec{v}_t,\phi\right) = \int_{\mathbb{R}^2} \frac{\mu(r,t)}{-\pi t}\mathrm{d}r\mathrm{d}t$$

□

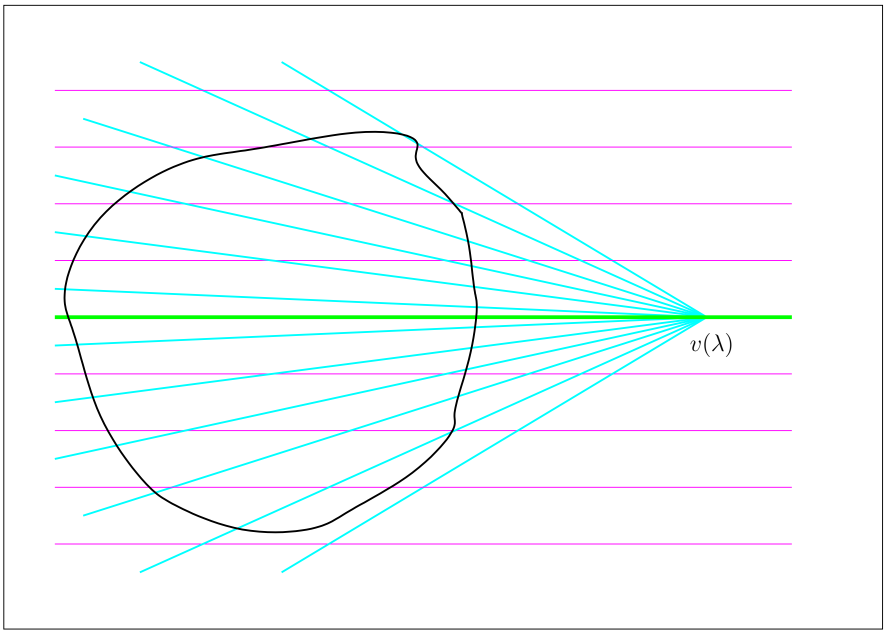

Figure 2.5: Parallel Fan Beam Hilbert Equality.

From theorem 2.5 we can derive a condition for the reconstruction of a ROI

Condition 2.3.1 (Data conditions for the reconstruction of a ROI)**.** A open region of interest can be reconstructed provided each line passing through the region of interest intersects the vertex path.

We can also propose an algorithm for the reconstruction of $\vec{x}$ in the ROI satisfying the previous condition.

1. Hibert filtering and rebinning step: compute $p_H(\phi, s)$ at $s = \vec{\theta}(\phi) \cdot \vec{x}$ from $d(t, \phi)$ such that $\vec{\theta}(\phi) \cdot \vec{x} = \vec{\theta}(\phi) \cdot \vec{v}_t$.
2. Derivation step: compute $\frac{\partial p_H}{\partial s}(\phi, s)_{arroverts=\vec{\theta}(\phi)\cdot\vec{x}}$ and back project.

Virtual Fan Beam Projections inversion formula

2.3.2 Differentiated Backprojection with Hilbert filtering

Introduction

The idea of the Differentiated Backprojection is to compute the Hilbert transform of μ along a direction $\vec{\alpha}$ from the back projection of the devivation of the projection. μ is then reconstructed from the inversion of the Hilbert transform.

Directional Hilbert transform

We consider a direction $\vec{\alpha} \in \mathcal{S}^1$ and and a function $\mu : \mathbb{R}^2 \to \mathbb{R}$; we define $H_{\vec{\alpha}}$ the Hilbert transform in the direction $\vec{\alpha}$ by

$$\begin{array}{rcl} H_{\vec{\alpha}}\mu : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ \vec{x} & \longrightarrow & H_{\vec{\alpha}}\mu(\vec{x}) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \mu(\vec{x} - u\vec{\alpha})h(u)\mathrm{d}u = \int_{\mathbb{R}} \frac{\mu(\vec{x}-u\vec{\alpha})}{\pi u}\mathrm{d}u \end{array}$$

We have

$$H_{-\vec{\alpha}}\mu = -H_{\vec{\alpha}}\mu$$

We naturally extend the definition to $\vec{v} \neq 0$ by

$$H_{\vec{v}}\mu(\vec{x}) \stackrel{\text{def}}{=} H_{\frac{\vec{v}}{\|\vec{v}\|}}\mu(\vec{x})$$

Differentiated Backprojection

Differentiated Backprojection and Hilbert Transform

We consider the derivation according to the trajectory parameter t of the Fan Beam projection

$$\begin{aligned} d_D\left(\vec{v}(t), \phi\right) & \stackrel{\text{def}}{=} \frac{\partial}{\partial t} d\left(\vec{v}(t), \phi\right) \\ & = \frac{\partial}{\partial t} \int_0^{+\infty} \mu\left(\vec{v}_t + l\vec{\theta}(\phi)\right) \mathrm{d}l = \frac{\partial}{\partial t} \int_{+\infty}^{+\infty} \mu\left(\vec{v}_t + l\vec{\theta}(\phi)\right) \mathrm{d}l \\ & = \int_{+\infty}^{+\infty} \frac{\partial}{\partial t} \mu\left(\vec{v}_t + l\vec{\theta}(\phi)\right) \mathrm{d}l \\ & = \int_{+\infty}^{+\infty} \nabla\mu\left(\vec{v}_t + l\vec{\theta}(\phi)\right) \cdot \frac{\partial}{\partial t} v(t) \mathrm{d}l \end{aligned}$$

where we have use $v(t) \notin \Omega$. Now, we consider $\overset{<}{b}_{t_1,t_2}$ the back projection of d_D on $[t_1, t_2] \subset T$

$$\overset{<}{b}_{t_1,t_2}(\vec{x}) \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{t_1}^{t_2} \frac{1}{\|\vec{x} - \vec{v}_t\|} \int_{+\infty}^{+\infty} \nabla\mu\left(\vec{v}_t + l\vec{\theta}\left(\arg\left(\vec{x} - \vec{v}_t\right)\right)\right) \cdot \frac{\partial}{\partial t} v(t) \mathrm{d}l\mathrm{d}t \tag{2.6}$$

We remark that $\vec{\theta}\left(\arg\left(\vec{x} - \vec{v}_t\right)\right) = \frac{\vec{x}-\vec{v}_t}{\|\vec{x}-\vec{v}_t\|}$. In Eq. (2.6) we make the change of variables $1-u = \frac{l}{\|\vec{x}-\vec{v}_t\|}$ thus $\mathrm{d}t = \frac{\mathrm{d}l}{\|\vec{x}-\vec{v}_t\|}$ and thus $\vec{v}_t + l\frac{\vec{x}-\vec{v}_t}{\|\vec{x}-\vec{v}_t\|} = \vec{v}_t + (1-u) = \vec{x} - u\left(\vec{x} - \vec{v}_t\right)$. We obtain

$$\overset{<}{b}_{t_1,t_2}(\vec{x}) = \frac{1}{\pi} \int_{t_1}^{t_2} \int_{+\infty}^{+\infty} \frac{\partial}{\partial t} v(t) \cdot \nabla\mu\left(\vec{x} - u\left(\vec{x} - \vec{v}_t\right)\right) \mathrm{d}u\mathrm{d}t$$

Moreover we have

$$\nabla\mu\left(\vec{x}(1-u) + u - \vec{v}_t\right) \cdot \frac{\partial}{\partial t} v(t) = \frac{1}{u} \frac{\partial}{\partial t} \mu\left(\vec{x}(1-u) + u\vec{v}_t\right)$$

Thus we have

$$\begin{aligned}
\overset{<}{b}_{t_1,t_2}(\vec{x}) &= \frac{1}{\pi}\int_{+\infty}^{+\infty}\frac{1}{u}\int_{t_1}^{t_2}\frac{\partial}{\partial t}\mu\left(\vec{x}(1-u)+u\vec{v}_t\right)\mathrm{d}t\mathrm{d}u \\
&= \frac{1}{\pi}\int_{+\infty}^{+\infty}\frac{1}{u}\left(\mu\left(\vec{x}-u\left(\vec{x}-\vec{v}_{t_2}\right)\right)-\mu\left(\vec{x}-u\left(\vec{x}-\vec{v}_{t_1}\right)\right)\right)\mathrm{d}u \\
&= H_{\vec{x}-\vec{v}_{t_2}}\mu(\vec{x}) - H_{\vec{x}-\vec{v}_{t_1}}\mu(\vec{x})
\end{aligned}$$

We remark now that if $\vec{x}\in(\vec{v}_{t_1},\vec{v}_{t_2})$ then $\vec{x}-\vec{v}_{t_2} = k_2\left(\vec{v}_{t_1}-\vec{v}_{t_2}\right)$ with $k_2>0$ and $\vec{x}-\vec{v}_{t_1} = k_1\left(\vec{v}_{t_1}-\vec{v}_{t_2}\right)$ with $k_1<0$. Thus, if $\vec{x}\in(\vec{v}_{t_1},\vec{v}_{t_2})$ we have

$$\overset{<}{b}_{t_1,t_2}(\vec{x}) = H_{\vec{v}_{t_1}-\vec{v}_{t_2}}\mu(\vec{x}) + H_{\vec{v}_{t_1}-\vec{v}_{t_2}}\mu(\vec{x}) = 2H_{\vec{v}_{t_1}-\vec{v}_{t_2}}\mu(\vec{x})$$

If we now consider the back projection between t and t_1 and the back projection between t and t_2 typically for $t_1<t<t_2$ with t_1 and t_2 such that $\vec{x}\in(\vec{v}_{t_1},\vec{v}_{t_2})$ we have

$$\overset{<}{b}_{t,t_1}(\vec{x})+\overset{<}{b}_{t,t_2}(\vec{x}) = H_{\vec{x}-\vec{v}_{t_1}}\mu(\vec{x})-H_{\vec{x}-\vec{v}_t}\mu(\vec{x})+H_{\vec{x}-\vec{v}_{t_2}}\mu(\vec{x})-H_{\vec{x}-\vec{v}_t}\mu(\vec{x}) = 2H_{\vec{v}_t-\vec{x}}\mu(\vec{x})$$

because $H_{\vec{x}-\vec{v}_{t_1}}\mu(\vec{x}) = H_{\vec{v}_{t_2}-\vec{v}_{t_1}}\mu(\vec{x}) = -H_{\vec{x}-\vec{v}_{t_2}}$ for $\vec{x}\in(\vec{v}_{t_1},\vec{v}_{t_2})$.

Inversion of the Finite Hilbert transform

DBP-H Inversion

2.3.3 Advances with DBP-H approaches

New Geometries

One Side Hilbert transform inversion

New appraoches for the interior problem

Appendix A

Numerics

A.1 Fourier Transform

For more information on Fourier analysis and numerical application, see [9] or [8].

A.1.1 DFT and Fourier coefficient of a function in $L^2_P(a)$

Let $L^2_P(a)$ be the set of periodic functions of period $a > 0$ such that $|f(t)|$ on a period. Such a function $f \in L^2_P(a)$ can be expanded in a Fourier serie

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n(f) e^{\frac{2i\pi nt}{a}}, \text{ where } c_n(f) = \frac{1}{a}\int_0^a f(t) e^{-\frac{2i\pi nt}{a}} dt.$$

The convergence of the serie $\sum_n c_n(f) e^{\frac{2i\pi nt}{a}}$ depends on the smoothness of f. Point wise convergence is obtained for bounded variation functions (Dirichlet's theorem). Uniform convergence is obtained for $\mathcal{C}^1$ function and for more regular functions the Fourier serie has a faster convergence.

A.1.2 Link between the Fourier Transform and the DFT

The Discrete Fourier Transform is linked to the approximation of the Fourier coefficients $c_n(f)$ of the Fourier series of $f \in L^2_P(a)$. More precisely let $\mathbf{f} \in \mathbb{C}^N$ a regular N-point discretization of f on $[0, a[$, i.e $f_l = f(t_l)$ with $t_l = l\frac{a}{N}, l = 0, \ldots, N-1$, and let $\mathbf{F} = DFT(\mathbf{f})$ be the Discrete Fourier Transform of $\mathbf{f}$ then $F_n \approx c_n(f), n = 0, \ldots, N/2 - 1$ and $F_n \approx c_{n-N}(f), n = N/2, \ldots, N-1$.

Let suppose now that $f \in L^2(\mathbb{R})$ is a real variable complex function of compact support contained in $[0, a]$. Then $f \in L^1(\mathbb{R})$ and we have

$$\hat{f}(\sigma) = \int_{\mathbb{R}} f(s) e^{-2i\pi\sigma s} ds = \int_0^a f(s) e^{-2i\pi\sigma s} ds.$$

Let $f_{P(a)}$ be the periodization of f of period a such that $f_{P(a)}(s) = f(s \mod a)$, see Fig. A.1. We have

$$\hat{f}(\sigma_n) = ac_n\left(f_{P(a)}\right) \text{ where } \sigma_n = \frac{n}{a}$$

thus the DFT can be considered a discretization of the Fourier Transform for function of compact support. If the support of f is contained in a set of length $a > 0$ and f is discretized on N points on the set $[0, a]$ then $\frac{1}{a}$ is the sampling step of the DFT in the Fourier, i.e., $F_n \approx \frac{1}{a}\hat{f}\left(\frac{n}{a}\right), n = 0, \ldots, N/2-1$ and $F_n \approx \frac{1}{a}\hat{f}\left(\frac{n-N}{a}\right), n = N/2, \ldots, N-1$.

The generalization to functions f of compact support contained in $[b, c]$, where $a \stackrel{\text{def}}{=} c - b > 0$ is simple.

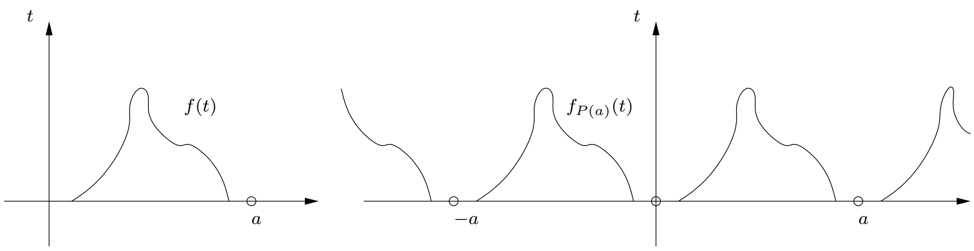

Figure A.1: Periodization $f_{P(a)}$ (right), of period a, of f a function of compact support contained in $[0, a]$ (left).

A.2 2D

A.2.1 Computing filters

The Hibert Filter

For numerical application we consider a frequency cut-off version of the Hibert filtering (Low Pass approximation of the Hiber filter). We define h_{Hc} such that $\hat{h_{Hc}}(\sigma) = \chi_{[-c,c]}(\sigma)(-i\text{sgn}(\sigma))$ where $c > 0$ is a cut-off frequency parameter. Then

$$h_{Hc}(s) = \int_{-c}^{c} -i\text{sgn}(\sigma)\mathrm{e}^{2\mathrm{i}\pi\sigma\mathrm{s}}\mathrm{d}\sigma = \frac{1-\cos(2\pi\mathrm{cs})}{\pi\mathrm{s}}.$$

We remark that if we choose $c = \frac{1}{2h}$ then whith $s_l = lh, l \in \mathbb{Z}$:

$$h_{Hc}(s_l) = \frac{1-\cos(l\pi)}{l\pi h} = \begin{cases} \frac{2}{l\pi h} & \text{for } l \text{ odd} \\ 0 & \text{for } l \text{ even} \end{cases}$$

Glossary

$L_P^p(a)$ Space of periodic functions f of period $a > 0$ such that the integral of $|f|^p$ exists with $p > 0$. 12

Acronyms

FBP Filtered Back Projection. 13

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